SELF-SIMILAR PROCESSES IN COMMUNICATIONS NETWORKS

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Abstrakt

Recent traffic measurements in corporate LANs, Variable-bit-rate video sources, ISDN control-channels and other communication systems, have indicated traffic behaviour of self-similar nature. This paper reviews and discusses briefly the known definitions and properties of second-order self-similar discrete-time processes and supplements them with some more general conditions of self-similarity. A model for ATM cell traffic is presented and self-similarity conditions of this model are found.

1. INTRODUCTION

Long-memory and self-similar processes have been studied since the middle of this century. They were discovered experimentally and introduced mathematically in a remarkably large number of fields, such as agronomy, astronomy, chemistry, economics, engineering, enviromental sciences, geosciences, hydrology, mathematics, physics, and statistics. Pioneering mathematical work on them was done by Kolmogorov [9] and Mandelbrot [14].

Recently, these processes were considered in modeling cell traffic in modern communications networks. In particular, *second-order* self-similar processes were used to model telecommunication traffic in high-resolution Ethernet local-area networks, wide-area networks and also for variable-bit-rate video traffic. This was motivated by experimentally observed long-range dependence of traffic data and "burstiness" of traffic streams across an extremely wide range of time scales: traffic spikes riding on longer-term ripples, that in turn ere riding on still longer term swells, etc. [11], [12], [5], [16].

Inspite the growing interest in this new telecommunication subject, there are no books which consider in detail the *second-order* self-similar processes and explain their properties. A lean literature on this *second-order* topic is scattered over years, journals, and papers. Also, the main attention of even existing literature oriented to non-telecommunications applications is focused on the self-similar processes which are continuous both in values and in time (see the list of about 800 references in [1] and the list of 420 references in [22]) whereas cell and packet commutications networks demand traffic models which are discrete both in values and in time.

Additionally, there is some unpreparedness of second-order self-similar processes for immediate use. This is related to the fact that inspite the existence of concepts of exact and asymptotic second-order self-similarity, there are no commonly accepted unique definitions of such processes.

Leland et al. [11], [12] suggested the use of parsimonious models for self-similar traffic. Such models (for example, having just three parameters such as mean, variance, and the Hurst parameter) form a rather wide set. Generally speaking, more parsimonious models lead to more wide sets. The problem is to take from such a set a relatively narrow subset of models which are both mathematically tractable and physically reasonable.

The main object of the present paper is to review and discuss briefly the known definitions and properties of second-order self-similar discrete-time processes, to supplement them with some more general conditions of self-similarity, to present a model for ATM cell traffic, and, finally, to find the conditions of model self-similarity.

Section 2 contains definitions of exactly and asymptotically second-order self-similar processes, which we accept. The most essential second-order properties of these processes are presented. A novelty is in presentation of some unknown proofs and properties as well as in presentation of all these properties in one paper. A comparision of different definitions is done with discussion and comments.

Section 3 gives a model for ATM cell traffic, the necessary and sufficient conditions for its exact self-similarity and a sufficient condition for its asymptotical self-similarity. The conditions are more general than obtained earlier ; they contain the known conditions as special cases. We reference the previous papers which are particularly relevant to the model and also discuss some other known models linked to our model only objectivelly.

The proofs of our results are placed in Appendices A-D. In this presentation, we need to use the concepts of the Karamata slow- and regular-variation theory. The definitions of slowly and regularly varying functions and sequences are given in Appendix E. For other used facts of the theory, we refer to [2].

2. SECOND-ORDER SELF-SIMILAR DISCRETE-TIME PROCESS

This Section contains the definitions of exactly and asymptotically second-order self-similar processes. The class of exactly self-similar processes is too narrow for modeling a real network traffic. Usually, it is exploited for theoretical purposes. As to the class of asymptotically self-similar processes, it is broad enough for communication network applications.

The most important modern communications networks are discrete in nature since they use binary and byte signals and measures and frame, packet, and cell transmissions. Taking this into account, we focus in this paper on traffic models only discrete in values and in time. However, the definitions and considerations in Section 2 hold for continuous-value and discrete-time processes as well.

For exactly self-similar process, some of its main properties are reviewed and commented. The definitions of exactly and strictly self-similar processes are compared. These considerations are known and can be found in [6], [9], [14] and [17] - [20].

For an asymptotically self-similar process, we give a necessary and sufficient condition of its self-similarity in terms of variance of its averaged version. Also, we give a new sufficient condition of self-similarity in terms of correlation coefficient of the process itself.

We begin with the introduction of a semi-infinite segment of a second-order-stationary real-number stochastic process $X = (X_1, X_2, \ldots)$ of discrete argument (time) $t \in I_1 \hat{=} \{1, 2, \ldots\}$.

Denote $m = \mathrm{E}X_t < \infty$ and $s^2 = \mathrm{var}\, X_t < \infty$, the mean and the variance of X_t respectively. Denote

$$r(k) = \frac{\mathrm{E}(X_{t+k} - m)(X_t - m)}{s^2}, \qquad k \in I_0 = \{0,1,2,...\},$$

the correlation coefficient of process X and denote by *f(l)* its spectral density. The mean m, the variance s^2, and the correlation coefficient $r(k)$ do not depend on time t, and $r(k) = r(-k)$.

2. 1. Second-order exact self-similarity. The subsection opens with two definitions of exact second-order self-similarity. They are equivalent (see Theorem 1 below). The first definition, A, is more appropriate to use for proving the exact self-similarity of a specific process and, also, it is more convenient for generalization of exact self-similarity to asymptotical one. The second definition, B, is more suitable for understanding burstiness of self-similar processes and for justification of term self-similarity.

Definition A. A process X is called *exactly second-order self-similar (es-s)* with parameter $H = 1 - \frac{b}{2}$, $0 < b < 1$ if its correlation coefficient is

$$r(k) = \frac{1}{2}[(k+1)^{2-b} - 2k^{2-b} + (k-1)^{2-b}] \hat{=} g(k), \quad k \in I_1 = \{1,2,...\}. \qquad (2.\ 1)$$

The function $g(k)$ can be written as $g(k) = d^2(k^{2-b})$ in terms of the central second difference operator $d^2(f(x))$ applied to a function $f(x) = x^{2-b}$. Note that $d(f(x)) = f(x + \frac{1}{2}) - f(x - \frac{1}{2})$. In (2. 1) and in the sequel,the symbol $\hat{=}$ means the equality by definition.

For the presentation of a second definition, we need to introduce the averaged (over blocks of length *m*) process $X^{(m)}$,

$$X^{(m)} = (X_1^{(m)}, X_2^{(m)}, \ldots),$$

$$X_t^{(m)} = \frac{1}{m}(X_{tm-m+1}^{(m)} + \ldots + X_{tm}^{(m)}), \quad m, t \in I_1,$$

its correlation coefficient $r_m(k)$, and its variance V_m. The averaged process $X^{(m)}$, as well as the normalized process $\breve{X}^{(m)}$ which will come later, play an important role in getting an insight to self-similarity of process *X*.

Definition B. A process X is called *exactly second-order self-similar (es-s)* with parameter $H = 1 - \frac{b}{2}$, $0 < b < 1$ if $r_m(k) = r(k), \quad k \in I_0, \quad m \in I_1.$

Before commenting, we present the most essential properties of es-s processes. The properties are given by the following

Theorem 1. *For a process X and* $0 < b < 1$, *the following are equivalent:*

(a) X is es-s in definition A, *i. e.* $r(k) = g(k)$,

(b) $V_m \hat{=} \mathrm{var}\, X_t^{(m)} = s^2 m^{-b}, \quad m \in I_2 = \{2,3,\ldots\},$

(c) $f(l) = c|e^{2pil} - 1|^2 \sum_{l=-\infty}^{\infty} \frac{1}{|l + l|^{3-b}}, \quad -\frac{1}{2} \le l \le \frac{1}{2},$

(d) X is es-s in definition B, *i. e.* $r_m(k) = r(k), \quad k \in I_0, \quad m \in I_1.$

Theorem 1 shows the equivalence of the above two definitions and that a process *X* with $r(k) = g(k)$ does not change its second-order characteristics, correlation and spectrum shape, with averaging over blocks of any length *m*.. This is why *X* is called self-similar. (The claim that (*a*) follows from (*d*) was published by Cox [6] but without proof, therefore we prove it at the end of this subsection.) The significance of function $g(k)$ is in the fact that it gives a non-degenerate correlational structure of limiting averaged process.

Since $g(k) \sim \frac{1}{2}(2-b)(1-b)k^{-b} = H(2H-1)k^{-b}$, $k \to \infty$, the es-s process has a heavy-tailed and even unsummable autocovariance function and correlation coefficient, $\sum_{k=0}^{\infty} r(k) = \infty$. (Here and below, $f(x) \sim h(x)$ means $[f(x)/h(x)] \to 1$ as $x \to \infty$.) This relates the es-s processes with the *long-range dependence (l-rd)* processes. The last were defined (see [1], [6]) as processes which have $r(k) \sim ck^{-b}, 0 < b < 1$, where *c* is a constant. Thus, we see that any es-s process is l-rd.

What is the rôle of parameter b? First, $b > 1$ shows that X is the *short-range dependence (s-rd) process* which has $V_m \sim cm^{-1}$ as $m \to \infty$, summable $r(k)$, and uncorrelated $X_t^{(m)}$ as $m \to \infty$, whereas $0 < b < 1$ shows that X is lr-d process. Second, when $0 < b < 1$, a value of b shows a level of long-range dependency in X, a higher b corresponds to higher dependency in X.

Along with *second-order exactly self-similar* processes, just *self-similar* processes, which for greater terminological difference we call *strictly self-similar (ss-s)* processes, are even more widely known . We recall their definition and then compare es-s and ss-s processes.

Definition. A strict-sense stationary process $X = (X_1, X_2, \ldots)$ is called *strictly self-similar (ss-s)* with parameter $H = 1 - \frac{b}{2}$, $0 < b < 1$ if $m^{1-H} X^{(m)} \overset{\text{dis}}{=} X$ (or, equivalently, $\breve{X}^{(m)} \overset{\text{dis}}{=} X$) where $\breve{X}^{(m)} = (\breve{X}_1^{(m)}, \breve{X}_2^{(m)}, \ldots)$, $\breve{X}_t^{(m)} = \frac{1}{m^H}(X_{tm-m+1} + \ldots + X_{tm})$, and $\overset{\text{dis}}{=}$ means equality in the sense of finite-dimentional distributions.

Our comparision of es-s and ss-s processes is expressed in terms of the following relations :

$$[r_m(k) = \breve{r}_m(k) = r(k)] \Leftarrow [\breve{X}^{(m)} \overset{\text{dis}}{=} X] \Rightarrow [\text{var}\, \breve{X}_t^{(m)} = s^2 \hat{=} \text{var}\, X_t], \qquad (2.2)$$

$$[r_m(k) = \breve{r}_m(k) = r(k)] \Leftarrow [\text{var}\, \breve{X}_t^{(m)} = s^2] \Leftrightarrow [r(k) = g(k)] \Leftrightarrow [\text{var}\, X_t^{(m)} = s^2 m^{-b}] \quad (2.3)$$

where $\breve{r}_m(k)$ is the correlation coefficient of the normalized process $\breve{X}^{(m)}$ and the relations (2.2) and (2. 3) are valid for all $k, m \in I_1$. (We note that the case $m = 1$ is trivial and the case $k = 0$ is not covered since $r_m(0) = \breve{r}_m(0) = r(0) = 1$ and $g(0)$ is not determined.)

These relations show that any ss-s process has $r(k) = g(k)$. It means in particular that if *X* is ss-s, then it is es-s. The opposite statement is not true, i. e. "*X* is es-s" does not imply "*X* is ss-s". However if *X* is Gaussian es-s with $\mathrm{E}X_t = 0$, then it is ss-s. Also, they prove the following implications for the Gaussian process *X* with $\mathrm{E}X_t = 0$:

[*X* is zero-mean Gaussian and ss-s]$\Rightarrow$ (2. 4)

$\Rightarrow\{[r_m(k) = \breve{r}_m(k) = r(k) = g(k)],[\text{var } \breve{X}_t^{(m)} = s^2],[\text{var } X_t^{(m)} = s^2 m^{-b}]\}$,

[*X* is zero-mean Gaussian and $r(k) = g(k)$]$\Rightarrow$ (2. 5)

$\Rightarrow\{[], [r_m(k) = \breve{r}_m(k) = r(k) = g(k)], [\text{var } \breve{X}_t^{(m)} = s^2], [\text{var } X_t^{(m)} = s^2 m^{-b}]\}$,

[*X* is zero-mean Gaussian and $\text{var } \breve{X}_t^{(m)} = s^2$ (or equivalently, $\text{var } X_t^{(m)} = s^2 m^{-b}$)]$\Rightarrow$

$\Rightarrow\{[\breve{X}^{(m)} \overset{\text{dis}}{=} X], [r_m(k) = \breve{r}_m(k) = r(k) = g(k)]\}$, (2. 6)

[*X* is zero-mean Gaussian and $\breve{r}_m(k) = r(k)$]$\Rightarrow[\breve{X}^{(m)} \overset{\text{dis}}{=} X]\Rightarrow$ (2. 7)

$\Rightarrow\{ [r_m(k) = \breve{r}_m(k) = r(k) = g(k)], [\text{var } \breve{X}_t^{(m)} = s^2], [\text{var } X_t^{(m)} = s^2 m^{-b}]\}$

where $k, m \in I_1$.

The implications (2. 4) - (2. 7) show the equvalence of the following assertions: [Zero-mean Gaussian *X* is ss-s], $[r_m(k) = \breve{r}_m(k) = r(k)]$, [var $\breve{X}_t^{(m)} = s^2$], and $[r(k) = g(k)]$.

Also, it follows that a solution $r(k) = g(k)$ to the functional equation $r_m(k) = r(k),\ k, m \in I_1$ is unique in the class of correlation coefficients $r(k)$,i.e. in the class of non-negative definite functions. Since a proof of this claim which also was included in Theorem 1 was not published, we prove it here. Namely, (2. 4) and (2. 5) give that a zero-mean Gaussian *X* is ss-s iff $r(k) = g(k), k \in I_1$. Since $r_m(k) = \breve{r}_m(k),\ k, m \in I_1$, (2. 7) gives that if a zero-mean Gaussian process *X* has $r_m(k) = r(k),\ k, m \in I_1$, it is ss-s. From these two sentences, we get directly the uniqueness of the solution $r(k) = g(k)$ to the equation $r_m(k) = r(k),\ k, m \in I_1$.

A ss-s process can not have a non-zero mean , whereas an es-s process is allowed to have such a mean since the restriction is imposed only on its correlation coefficient or autocovariance function. Moreover, if X is a positive and non-degenerated process, neither X nor X -m can be ss-s.

2. 2. Second-order asymptotical self-similarity. There are different definitions of asymptotical self-similarity. We bring that which we accept for this paper and also we mention and comment on other definitions. Here we use the same notations as above.

Definition. A process X is called *asymptotically second-order self-similar (as-s)* with parameter $H = 1 - \dfrac{b}{2}$, $0 < b < 1$ if

$$\lim_{m\to\infty} r_m(k) = g(k), \quad k \in I_1. \tag{2. 8}$$

Thus, X is as-s if after averaging over blocks of length m and with $m \to \infty$, its correlational structure becomes identical to that of es-s process (not to that of X itself (!) as in the different definitions [X is as-s if $r(k) \to L(k)k^{-b}, k \to \infty$ and $r_m(k) \to r(k), m \to \infty$, for large enough k where $L(k)$ is a slowly varying function] accepted in [12] and [X is as-s if $r_m(k) \to r(k), m \to \infty, k \in I_1$] accepted in [19]). In other words, if $X^{(m)}$ goes to be es-s as $m \to \infty$, then X is as-s. The presented definition is due to Cox [6]. The as-s processes in definitions of [12] and [19] are discussed at the end of this subsection.

Evidently, an es-s process is as-s.

The following theorem gives a necessary and sufficient condition of as-s for X to be as-s in terms of variance, V_m, of averaged process $X^{(m)}$ and also, gives a sufficient condition in terms of correlation coefficient, $r(k)$, of process X itself.. The theorem uses the concept of slow and regular variation (see Appendix E).

Theorem 2. *For a process X and* $H = 1 - \frac{b}{2}$, $0 < b < 1$, *the following are equivalent:*

(e) X is as-s, i. e. (2. 8),

(f) $\frac{V_{km}}{V_m} \sim k^{-b}$, *integer* $m \to \infty, k \in I_1$.

The asymptotical equations

(g) $V_m \sim L(m)m^{-b}, \quad m \to \infty, \quad m \in I_1$

and

(h) $r(k) \sim s^2 H(2H-1)L(k)k^{-b}, \quad k \to \infty, \quad k \in I_1$

(where $L(x) > 0$ *in (g) is a slowly varying function (svf) and* $L(x)$ *in (h) is the same as in (g)) are equivalent and each implies (e) and (f).*

Proof is given in Appendix A. ∴

The asymptotical equation (*f*) is just a definition of the *index* $(-b)$ *regularly varying sequence (rvs)* V_m *with integer variable* (see Appendix E). Thus, Theorem 2 states that the asymptotical self-similarity of X is equivalent to the regular variation of variance of $X^{(m)}$.

We stress that rvs with integer variable does not necessary behave as a rvs with continuous variable.

Each of the conditions (*g*) and (*h*) is sufficient for X to be as-s. They contain a slowly varying function (svf) used at positive integer points m. The condition (*g*) requires the variance V_m to be regarded as a rvf . Similarly, (*h*) requires the same from the correlation coefficient $r(k)$.

The condition (*h*) is sometimes more convenient then (*f*) and (*g*) since (*f*) and (*g*) are expressed via the characteristics of the average process, $X^{(m)}$, whereas (*h*) is expressed via the correlation coefficient of process X itself. This condition (*h*) is more general than the sufficient condition of asymptotical self-similarity, $r(k) \sim ck^{-b}, \quad k \to \infty, \quad c = \text{const}$, given in [20].

According to (*h*), each l-rd process in definitions of [1], [6] is as-s.

At the end, we remind that there exists a concept of *strictly asymptotically self-similar (sas-s)* process. A process X is sas-s if $\breve{X}^{(m)} \overset{\text{dis}}{=} X$ holds as $m \to \infty$. A sas-s process is not as-s in definition (2. 8) but it is as-s in definition given in [12] and [19]. If a different definition of sas-s process is accepted, namely, "X is sas-s if $\breve{X}^{(m)}$ is ss-s as $m \to \infty$", then we get the opposite, namely, that a sas-s process is as-s in definition (2. 8) but not as-s in definition given in [12] and [19].

Now, we note that the set of correlation coefficients $r(k)$ for which as-s process in definition of [12] exist, contains only one function $r(k) = g(k)$. It is so since if $r(k) \to L(k)k^{-b}$ then $r_m(k) \to g(k)$ according to Theorem 2. Thus, the class of as-s processes in definition of [12] coincide with the class of es-s processes.

Also, it is easy to note that the intersection of the class of as-s processes in definition of [19] and the class of as-s processes in the Cox definition accepted here contains only one process, namely, es-s process. Thus, the l-rd processes satisfying (h) except es-s process are not in the class of as-s processes in definition of [19]. In return, the second-order pure noise ($r(k) = 0, k \in I_1$) being the limit of averaged short-range dependence processes with $V_m \sim cm^{-1}, m \to \infty$ is as-s in definition of [19] but not as-s in the Cox definition.

3. TRAFFIC MODEL AND CONDITIONS OF ITS SELF-SIMILARITY

In this section, a model of cell traffic Y is described. The model was suggested in [21], the special cases of it were presented in [7], [10], [13] and [20]. We give the conditions for exact and for asymptotic self-similarity of the traffic Y. The analogous conditions in [7], [13], [20], and [21] are less general and less explicit than presented here. At the end, other known models of self-similar traffic are reviewed.

3. 1. Cell traffic model. A considered traffic Y is assumed to be a stream of cells. The cells have an equal length taken here as a time unit. The cells are assigned to sources so the traffic is an aggregation of cells generated by sources. To present a precise traffic structure, we begin with the source model.

The sources are numbered by s. A source s starts to generate cells at time denoted by w_s (the source numbering is ordered such that $w_s \le w_{s+1}$). The moment w_s is called the arrival epoch of source s. The source s generates $q_s(i) \in I_0$ cells at time $w_s + i - 1$ in time interval $w_s, \dots, w_s + t_s - 1$. The sequence $(q_s(1), \dots, q_s(t_s))$ is called the active period of source s and t_s is called the lengh of the source active period. Before time w_s and after time $w_s + t_s - 1$, the source s does not generate any cells, $q_s(i) = 0$ for $i < w_s$ and for $i \ge w_s + t_s$. Thus, $q_s(t - w_s + 1)$, $t \in I_1$ is the sequence of numbers of cells generated by source s at successive time moments.

The special cases of active period can be, for example,

* A constant $q_s(i) = R \in I_1, 1 \le i \le t_s$,
* A random constant $q_s(i) = R$ with the constant R depending on t_s, i. e. $R = R(t_s)$,
* The i. i. d. $q_s(i)$ taking on values 0 and 1 with probabilities p_0 and p_1 respectively,
* The i. i. d. $q_s(i)$ taking on values from $\{0, 1, \dots, k\}$ with binomial distribution or from I_0 with geometrical, Poissonian or some given distribution,
* Any Markov, semi-Markov or other well known sequences of $q_s(i)$.

A time t can be an arrival epoch for several sources. Let x_t be the number of sources with arrival epochs being equal to t.

The considered traffic $Y = (\ldots, Y_{-1}, Y_0, Y_1, \ldots)$ is an aggregation of cells generated by different sources,

$$Y_t = \sum_s q_s(t - w_s + 1), \quad t \in I_{-\infty} = \{\ldots, -1, 0, 1, \ldots\}. \tag{3. 1}$$

It means that Y_t is the total number of cells generated by all active sources at time t.

It is assumed that

(1^0) The active periods $(q_s(1), \ldots, q_s(t_s))$ are i. i. d. for different s, in other words, $(t_s, (q_s(1), \ldots, q_s(t_s)))$ are i. i. d. for different s. An active period $(q_s(1), \ldots, q_s(t_s))$, conditional on $t_s = l$, is a segment of a second-order-stationary non-negative integer stochastic process generally depended on l.

(2^0) The numbers of source arrivals, x_t, $t \in I_{-\infty}$, are independent and they are identically Poisson distributed $\Pr\{x_t = k\} = e^{-l} \dfrac{l^k}{k!}$ where $0 < l < \infty$ is the parameter of the Poisson distribution, $l = \mathrm{E} x_t$.

(3^0) The active periods $(q_s(1), \ldots, q_s(t_s))$[or $(t_s, (q_s(1), \ldots, q_s(t_s)))$] are mutually independent of numbers x_t and epochs w_s. The numbers x_t are mutually independent of epochs w_s.

This concludes the specification of cell traffic model Y which we consider. In [21], a special case of this model is considered, in which $(q_s(1), \ldots, q_s(t_s))$, conditional on $t_s = l$ with different s, are the segments of a second-order-stationary non-negative integer stochastic process independent of l. (We mean the model $Y^{(4)}$ in [21]. By reason of this independence on l, the model $Y^{(4)}$ in [21] is not able to cover the modes $Y^{(2)}$ and $Y^{(3)}$ there as special cases.) A special case of the model, namely, $q_s(i) = 1$ for all i inside of active period and for all s, was considered earlier in [7], [10], [13], and [20].

$Y = (\ldots, Y_{-1}, Y_0, Y_1, \ldots)$ can be interpreted as the random point marked process of discrete time, for which the points are s[or w_s] and the mark of point s is $(t_s, (q_s(1), \ldots, q_s(t_s)))$ or $(q_s(1), \ldots, q_s(t_s))$.

With the Poisson process splitting argument, the process Y can be splitted into an infinite number of independent processes $Y(l), l \in I_1$. An individual process $Y(l)$ has the same structure as process Y but with given active-period length $t_s = l$ for all its sources and with the number of sources arrived at t equal to $x_{t,l}$, the Poisson random variable with parameter $l_l = \mathrm{E} x_{t,l} = l \Pr\{t = l\}$ where t is the generic symbol for t_s in process Y. This means

$$Y = \sum_{l=1}^{\infty} Y(l), \quad x_t = \sum_{l=1}^{\infty} x_{t,l}, \quad \sum_{l=1}^{\infty} l_l = l \tag{3. 2}$$

where $x_{t,l}$ are independent for different (t, l).

It is easy to imagine the splitting as following. Upon its arrival, each source is sent to process $Y(l)$ with probability $\Pr\{t_s = l\}$ independently of each other source and independently of the arrival epochs. If a source s is sent to $Y(l)$, it gets $t_s = l$ and has the taken-from-process- Y conditional-on- $t_s = l$ distribution of $(q_s(1),\ldots,q_s(t_s))$ as unconditional distribution of its active period in process $Y(l)$.

In the next subsection, we use the splitting of process Y to obtain the conditions of its self-similarity.

<u>3. 2. Conditions for self-similarity of traffic *Y*.</u> Our aim here is to get the conditions under which the introduced in 3. 1 cell traffic Y is self-similar. Since second-order self-similarity is defined in terms of correlation coefficient, we first need to find the expressions for mean and autocovariance function of process Y and then, with an expression for $r(k)$, to find the desired conditions of exact and asymptotic self-similarity of Y. The conditions for special cases of Y found in previous papers can be deduced easily from the conditions given here.

Consider a process Y. Consider its generic source active period $(q(1),\ldots,q(t))$, conditional on $t = l$. It is a segment of second-order-stationary discrete-time process denoted here as $q(t) = (\ldots,q(-1),q(0),q(1),\ldots)$. The probabilistic characteristics of $q(t)$ depend on l. Denote $m^{(l)} \mathrel{\hat{=}} \mathrm{E}q(t)$, the mean, and $B^{(l)}(k) \mathrel{\hat{=}} \mathrm{E}q(t)q(t+k)$, $k \in I_{-\infty}$, the autocorrelation function of process $q(t)$. We assume that $0 < m^{(l)} < \infty$ and $0 < B^{(l)}(k) < \infty$ to avoid singularities.

Theorem 3. *The mean and autocovariance function of process Y are*

$$\mathrm{E}Y_t = l \sum_{l=1}^{\infty} l \Pr\{t = l\} m^{(l)}, \tag{3. 3}$$

$$w(k) \mathrel{\hat{=}} \mathrm{cov}\{Y_t, Y_{t+k}\} = l \sum_{l=k+1}^{\infty} (l-k) \Pr\{t = l\} B^{(l)}(k), \quad k \in I_0 . \tag{3. 4}$$

Proof is given in Appendix B. ∴

There is a useful corollary of Theorem 3, which gives $\Pr\{t \geq l\}$ in terms of l, $w(k)$ and $B^{(l)}(k)$ in the case when $B^{(l)}(k)$ does not depend on l.

Corollary. *If $B^{(l)}(k) = B(k)$, where $B(k)$ does not depend on l, then*

$$\Pr\{t \geq k+1\} = \frac{w(k)}{l\,B(k)} - \frac{w(k+1)}{l\,B(k+1)} .$$

Theorem 3 and its corollary were proved in [20] and [21] in special cases of autocorrelation function $B^{(l)}(k)$.

The next theorem gives the conditions which are necessary and sufficient for exact self-similarity of process Y. They are expressed in terms of active-period length distribution, $\Pr\{t = l\}$, mean, $m^{(l)}$, and autocorrelation function, $B^{(l)}(k)$, of process $q(t)$ dependent on l.

Theorem 4. (**1**) *The process Y is es-s with $H = 1 - \dfrac{b}{2}$, $0 < b < 1$ if and only if $\Pr\{t = l\}$, $m^{(l)}$ and $B^{(l)}(k)$ are such that*

$$\frac{\sum_{l=k+1}^{\infty} (l-k) \Pr\{t = l\} B^{(l)}(k)}{\sum_{l=1}^{\infty} l \Pr\{t = l\} B^{(l)}(0)} = \frac{1}{2} d^2 (k^{2-b}), \quad k \in I_1, \tag{3. 5}$$

$$\sum_{l=1}^{\infty} l \Pr\{t = l\} m^{(l)} < \infty . \qquad (3.\ 6)$$

(**2**)*The process* Y *with* $B^{(l)}(k) = B^{(l)}$ *(where* $B^{(l)}$ *does not depend on* k *) is es-s with* $H = 1 - \frac{b}{2}$, $0 < b < 1$ *if and only if* $\Pr\{t = l\}$, $m^{(l)}$ *and* $B^{(l)}$ *are such that*

$$\frac{\Pr\{t = k\} B^{(k)}}{\sum_{l=1}^{\infty} \Pr\{t = l\} B^{(l)}} = v(k), \qquad (3.\ 7)$$

$$\sum_{k=1}^{\infty} k v(k) \frac{m^{(k)}}{B^{(k)}} < \infty \qquad (3.\ 8)$$

where

$$v(1) \hat{=} \frac{3^{2-b} - 2^{4-b} + 7}{4 - 2^{2-b}}, \quad v(k) \hat{=} \frac{d^4(k^{2-b})}{4 - 2^{2-b}}, \quad k \in I_2 \qquad (3.\ 9)$$

and $d^4(f)$ *denotes the central fourth difference operator applied to a function* f,

$$d^4(k^{2-b}) = (k+2)^{2-b} - 4(k+1)^{2-b} + 6k^{2-b} - 4(k-1)^{2-b} + (k-2)^{2-b}, \; k \in I_2. \qquad (3.\ 10)$$

(**3**) *The process* Y *with* $m^{(l)} = m$, $B^{(l)}(k) = B(k)$ *(where* m *is a constant and* $B(k)$ *does not depend on* l *) is es-s with* $H = 1 - \frac{b}{2}$, $0 < b < 1$ *if and only if* $\Pr\{t = l\}$ *and* $B(k)$ *are such that*

$$\Pr\{t = 1\} = \frac{\dfrac{3^{2-b} - 2^{3-b} + 1}{B(2)} + \dfrac{4 - 2^{3-b}}{B(1)} + \dfrac{2}{B(0)}}{\dfrac{2 - 2^{2-b}}{B(1)} + \dfrac{2}{B(0)}}, \qquad (3.\ 11)$$

$$\Pr\{t = k\} = \frac{d^2\left(\dfrac{d^2(k^{2-b})}{B(k)}\right)}{\dfrac{2 - 2^{2-b}}{B(1)} + \dfrac{2}{B(0)}}, \quad k \in I_2 \qquad (3.\ 12)$$

where

$$d^2\left(\frac{d^2(k^{2-b})}{B(k)}\right) = \frac{(k+2)^{2-b} - 2(k+1)^{2-b} + k^{2-b}}{B(k+1)} - 2\frac{(k+1)^{2-b} - 2k^{2-b} + (k-1)^{2-b}}{B(k)} +$$

$$+ \frac{k^{2-b} - 2(k-1)^{2-b} + (k-2)^{2-b}}{B(k-1)}, \qquad k \in I_2.$$

The distribution (3. 11)-(3. 12) *has the finite mean*

$$\mathrm{E}\, t = \left[1 + (1 - 2^{1-b}) \frac{B(0)}{B(1)}\right]^{-1}. \qquad (3.\ 13)$$

(**4**) *The process* Y *with* $m^{(l)} = m$, $B^{(l)}(k) = B$ *(where* m *and* B *are some constants) is es-s with* $H = 1 - \frac{b}{2}$, $0 < b < 1$ *if and only if* $\Pr\{t = l\}$ *is such that*

$$\Pr\{t = k\} = v(k), \quad k \in I_1 \qquad (3.\ 14)$$

where $v(k)$ *is defined by* (3. 9). *The distribution* (3. 14) *has the finite mean*

$$\mathrm{E}\,t = \frac{1}{2 - 2^{1-b}}. \qquad (3.\ 15)$$

Proof is given in Appendix C. ∴

Note that the conditions of exact self-similarity in claims (**1**)-(**3**) of Theorem 4 are the functional equations, whereas the condition (3. 28) in claim (**4**) is an explicit expression for the distribution of active-period length t.

The distribution (3. 9) is a decreasing function of $k \in I_1$. Asymptotically, $v(k)$ decays as

$$v(k) \sim \frac{(b+1)b(b-1)(b-2)}{4 - 2^{2-b}} k^{-(b+2)}, \quad 0 < b < 1, \quad k \to \infty. \qquad (3.\ 16)$$

The distribution $v(k)$ is heavy-tailed. It has a finite mean but infinite variance. Numerically, $v(k)$ is illustrated by Tables 1 and 2.

Table 1. $v(k)$ for $b = 0.6$.

k	1	5	10	15	20
v(k)	0. 8078E+00	0. 6428E-02	0. 1000E-02	0. 3503E-03	0. 1513E-03

Table 2. $v(k)$ for $b = 0.2$.

k	1	5	10	15	20
v(k)	0. 5713E+00	0. 2028E-01	0. 4243E-02	0. 1444E-02	0. 1414E-02

In the less explicit form for $v(k)$, the functional equation (3. 7) was obtained by Cox [6] in case $B^{(l)} = 1$; also it was given in [20] for $B^{(l)} = B \in I_1$ and in [21] for monotonic $B^{(l)}$ with values in I_1.

Finally, we present the sufficient conditions of asymptotical self-similarity of process Y. Given the previous results on as-s, one would hypothesize that $\Pr\{t = l\}$ has to have a heavy-tail to provide as-s to Y. This hypothesis is not correct. It is enough to require a heavy-tailed decay of the product $\Pr\{t = l\}B^{(l)}(l)$ to assure as-s for Y. This is a claim of the following

Theorem 5. *The process* Y *is as-s with* $H = 1 - \frac{b}{2}$, $0 < b < 1$ *if* $\Pr\{t = l\}$, $m^{(l)}$ *and* $B^{(l)}(k)$ *are such that*

$$\Pr\{t = l\}B^{(l)}(l) \sim L(l)l^{-(b+2)}, \quad l \to \infty, \qquad (3.\ 17)$$

$$\sum_{l=1}^{\infty} \Pr\{t = l\}B^{(l)}(0) < \infty, \quad \sum_{l=1}^{\infty} l\Pr\{t = l\}m^{(l)} < \infty \qquad (3.\ 18)$$

where $L(x)$ *is a slowly varying function.*

Proof is given in Appendix D. ∴

Subject to some minor restrictions (3. 18), the theorem states in fact that, if the product $\Pr\{t = l\}B^{(l)}(l)$ is an index $-(b+2)$ rvf, then the process Y is as-s.

We feel that the sufficient conditions (3. 17)-(3. 18) are maybe necessary or quite near to them but what is true is still a question.

The condition of as-s, which is more restrictive than (3. 17) (with $L(x) = const$), was known before. It was obtained by Cox[6] in special case $B^{(l)} = 1$; also, it is in [13] for $B^{(l)} = B \in I_1$ and in [20] and [21] for less general traffic models.

Let us consider the traffic examples for which (3. 17) satisfies.

Example 1. Let $\Pr\{t = l\}$ be a Pareto-type distribution (it is heavy-tailed one),

$\Pr\{t = l\} \sim c_0 l^{-a-1}, \quad 1 < a < 2, \quad l \to \infty,$

where c_0 is a constant or let it be such that

$\Pr\{t = l\} \sim L(l) l^{-a-1}, \quad 1 < a < 2, \quad l \to \infty,$

then Y is as-s with $H = (3-a)/2$ if $m^{(l)} = const$ and $B^{(l)}(k) = B = const$. ∴

Example 2. Let $\Pr\{t = l\}$ be negative exponential ("light-tailed"),

$\Pr\{t = l\} \sim c_0 e^{-jl}, \quad l \to \infty$

where $j > 0$ is a constant, then Y is as-s with $H = 1 - \dfrac{b}{2}$ if $m^{(l)} = const$ and $B^{(l)}(l) = B^{(l)} \sim L(l) l^{-(b+2)} e^{jl}, \quad 0 < b < 1, \quad l \to \infty$. ∴

However, it is easy to see that if $\Pr\{t = l\} = 0, \; l > l_0$ for some finite l_0, then Y can not be as-s.

3. 3. Other known models. In presentation of self-similar traffic above, we referenced the papers which were particularly relevant to our model. There are also other mathematical models of self-similar traffic in communication networks. An extensive bibliographical guide with 420 references to self-similar traffic modeling for modern high-speed networks is given in [22]. Not pretending on any full survey, we would like to briefly mention some important self-similar models which although are in a distance from our model, nevertheless have a certain conceptual relation with it.

All observed models below have continuous time. They use the term "source" for a different mathematical object than it is in our model above. There, a source is an on/off sequence. The source j has mutually independent alternating silence periods of i. i. d. lengths S_{ij} (with S_j or S as generic) and active periods of i. i. d. lengths A_{ij} (with A_j or A as generic), $i \in I_1$.

Boxma [3], [4] considers a traffic which is a superposition of N sources. A source j constantly transmits at rate $R_j > 1$ when active, contributing $R_j A_{ij}$ volume to the traffic during its i-th active period. Several special cases are considered, namely, [$N = 1$, distribution of A is rvf, distribution of S is arbitrary], [N is any given, distribution of A_1 is rvf, all A_j, $2 \le j \le N$ are i. i. d. with negative exponential (nex) distribution or with exponential-tail distribution, distributions of all S_j are nex], and [$N = \infty$, all A_j are i. i. d. with rvf distribution, all S_j are i. i. d. with nex distribution]. The papers do not give any results on self-similarity of considered traffics. For a queue fed by each of these traffics, they present the steady-state distribution of infinite-buffer content at some specific time moments.

The models considered in these two papers and the approach used in getting buffer content results have a close conceptual similarity to those which are in [13] and [21]. Namely, in [13], a discrete-time analog of the traffic is considered , which is the superposition of N i. i. d.

sources. It is proved that if $N \to \infty$, $N / (\mathrm{E}A + \mathrm{E}S) = \text{const}$, $\Pr\{S \le t\} \to 0$ for any $t < \infty$, and the distribution of A does not depend on N and is rvf, then the traffic is as-s. Also, for a queue fed by this limitting traffic, the steady-state infinite-buffer content distribution (precisely, its moment generating function) at some specific time moments is found.

Previously, an on/off (or "packet train") source model was proposed by Jain and Routhier [8] for LAN traffic modeling. Willinger, Taqqu, Sherman, and Wilson [23] remedied some of the shortcomings of the packet train model and focused their attention on conditions of self-similarity of superposition of on/off sources. Considering index $-a_A$ and index $-a_S$ rvf distributions for A and S, respectivelly, $1 < a_A < 2$, $1 < a_S < 2$, $a = \min(a_A, a_S)$, and assuming that these distributions have probability densities or they are non-arithmetic, they prove that the N-source superposition reduced to zero-mean and integrated from 0 to Tt tends (in the sense of the finite-dimentional distributions) to the fractal Brownian motion (fBm) scaled by a factor $T^H N^{1/2} L^{1/2}(T)$ as first $N \to \infty$ and then $T \to \infty$, where $H = (3 - a)/2$ and $L(T)$ is svf related to the distributions of A and S.

The (Gaussian) fBm process is considered by Norros [15] as a model for ATM traffic.

These models are far not the same as our model given in 3. 1 but all models are motivated by the same traffic measurements in high-speed communication networks, which are referenced in the Introduction section of the present paper.

4. CONCLUSIONS

The different known definitions of exactly and asymptotically second-order self-similar processes were reviewed. A comparision of these definitions was done with discussion and comments. We presented the most essential second-order properties of self-similar processes in the Cox definitions accepted here. Some of our proofs and the presented properties are new.

We then gave a model for ATM cell traffic and the necessary and sufficient conditions for its exact self-similarity and a sufficient condition for its asymptotical self-similarity. The conditions are more general than obtained erlier; they contain the known conditions as special cases. The previous papers which are particular relevant to the model were referenced and the conceptual relations between models were briefly discussed.

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APPENDIX A: PROOF OF THEOREM 2

Proof of Theorem 2. We start with a proof that (e) implies (f). The proof is based on the known equation [24],

$$r_m(k)=\frac{1}{2V_m}[(k+1)^2 V_{(k+1)m}-2k^2V_{km}+(k-1)^2 V_{(k-1)m}]\equiv\frac{1}{2V_m}d^2(k^2V_{km}),\quad k,m\in I_1. \tag{A. 1}$$

Using (2. 8) and (A. 1) both for $k=1$, we get $(\frac{2V_{2m}}{V_m}-1)\to(2^{2-b}-1)$, integer $m\to\infty$ that gives (*f*) for $k=1$. Again using (2. 8) and (A. 1) both for $k=2$ this time, we get $\frac{1}{2}\left[3^2\frac{V_{3m}}{V_m}-2^3\frac{V_{2m}}{V_m}+\frac{V_m}{V_m}\right]\to\frac{1}{2}[3^{2-b}-2^{3-b}+1]$ that gives (*f*) for $k=2$.

For $k>2$, (*f*) can be proved easy by induction.

That (*f*) implies (*e*), follows directly from (*f*) and (A. 1).

Next, we show that (*g*) implies (*f*). It is evident since V_m satisfying (*g*) is rvs with continuous variable and can be regarded as rvf .

Now, we prove that (*g*) implies (*h*). Taking (A. 1) for $m=1$, we get

$$r(k)=\frac{1}{2s^2}[(k+1)^2V_{k+1}-2k^2V_k+(k-1)^2V_{k-1}]\sim$$
$$\sim\frac{1}{2s^2}[(k+1)^{2-b}L(k+1)-2k^{2-b}L(k)+(k-1)^{2-b}L(k-1)]\sim$$
$$\sim s^{-2}L(k)g(k)\sim s^{-2}H(2H-1)L(k)k^{-b},\quad k\to\infty$$

where the equation $L(k+1)/L(k)\to1,\ k\to\infty$ was used , which holds for a svf (but does not for a svs with integer variable!).

At last, we prove that (*h*) implies (*g*). We start with the equation [24],

$$V_m=s^2m^{-1}+2s^2m^{-2}\sum_{i=1}^{m}(m-i)r(i),\quad m\in I_1 \tag{A. 2}$$

According to (*h*), $r(k)$ can be extended on a continuous neibourhood of infinity, $[a,\infty)$, $a\geq1$, as $-b$ rvf, i. e. $r(x)\sim L(x)x^{-b}$, where $L(x)$ is locally bounded in $[a,\infty)$. (For the definitions and properties of r rvf and locally bounded function, see [2].) Here and below, we assume that a is an integer.

We need the following

Lemma A. 1. *If* $f(x)$ *is a* r *rvf, then*

$$\sum_{i=a}^{n}f(i)\sim\int_a^n f(x)dx,\quad n\to\infty. \tag{A. 3}$$

Proof. Denote $a_i\hat{=}f(i)$, $b_i\hat{=}\int_i^{i+1}f(x)dx$. Note that $a_i\sim a_{i+1}$, $i\to\infty$ (Lemma 1. 9. 6 in [2]) and $\int_a^n f(x)dx=\sum_{i=a}^{n-1}b_i$.

Now, let us show that $a_i\sim b_i$, $i\to\infty$. We use the inequalities $m_i\leq b_i\leq M_i$ where for $r>0$, $m_i=\inf(f(x): i\leq x)$, $M_i=\sup(f(x): x\leq i+1)$ and for $r<0$, $m_i=\inf(f(x): x\leq i+1)$, $M_i=\sup(f(x): x\geq i)$. By the Karamata theorem (Theorem 1. 5. 3 in [2]), $m_i\sim a_i$ and $M_i\sim a_{i+1}$, $i\to\infty$. It means $a_i\sim b_i$, $i\to\infty$.

To conclude Lemma proof, we need only to use the following two Lemmas A. 2 and A. 3 without additional comments. ∴

Lemma A. 2. *Let* $a_i > 0$ *and let* $a_i \sim a_{i+1}, i \to \infty$. *Then*
$\sum_{i=a}^{n-1} a_i \sim \sum_{i=a}^{n} a_i, n \to \infty$ or $a_n [\sum_{i=a}^{n} a_i]^{-1} \to 0, n \to \infty$.

Proof. For any $e > 0$, we choose $0 < d < 1$ and N_1 such that $(1 + d + d^2 + \ldots + d^{N_1 - 1})^{-1} < e$. Since $a_i / a_{i+1} \to 1,\ i \to \infty$, we can choose N_2 such that for $i > N_2,\ a_i / a_{i+1} > d$. Then for $n > N_1 + N_2 + a$,

$$a_n [\sum_{i=a}^{n} a_i]^{-1} < a_n [\sum_{i=a}^{n-N_1} a_i + a_n (1 + d + \ldots + d^{N_1 - 1})]^{-1} < (1 + d + \ldots + d^{N_1 - 1})^{-1} < e\ .\ \therefore$$

Lemma A. 3. *Let* $a_i > 0$ *and let* $a_i \sim b_i, i \to \infty$ *and* $\sum_{i=a}^{n} a_i \to \infty, n \to \infty$. *Then*

$$\sum_{i=a}^{n} a_i \sim \sum_{i=a}^{n} b_i, \quad n \to \infty.$$

Proof. For any $e > 0$, we choose $N_1 > a$ such that $|b_i - a_i| < \frac{e}{2} a_i,\ i > N_1$ (we can do it since $(b_i - a_i) a_i^{-1} \to 0$ when $b_i / a_i \to 1, i \to \infty$). Then we choose N_2 such that for $n > N_2 + a$, $\left|\sum_{i=a}^{N_1} (b_i - a_i)\right| / \sum_{i=a}^{n} a_i < \frac{e}{2}$ (we can do it since $\sum_{i=a}^{n} a_i \to \infty, n \to \infty$). For $n > N_2 + a$, we have

$$\left|[\sum_{i=a}^{n} b_i - \sum_{i=a}^{n} a_i] / \sum_{i=a}^{n} a_i\right| \leq \left|[\sum_{i=a}^{N_1} (b_i - a_i)] / \sum_{i=a}^{n} a_i\right| + [\sum_{i=N_1+1}^{n} |b_i - a_i|] / \sum_{i=a}^{n} a_i \leq$$
$$\frac{e}{2} + \frac{e}{2} [\sum_{i=N_1+1}^{n} a_i] / \sum_{i=a}^{n} a_i < e\ .\ \therefore$$

Now, we continue to prove Theorem 2. At first, we note that since for any finite integer $a > 0$, (h) and $r(i) \leq 1$ give $\sum_{i=1}^{a-1} r(i) < a$ and $\sum_{i=a}^{m} r(m) \to \infty, m \to \infty$, we have

$$\sum_{i=1}^{m} r(i) \sim \sum_{i=a}^{m} r(i), m \to \infty. \tag{A. 4}$$

Similarly,

$$\sum_{i=1}^{m} i r(i) \sim \sum_{i=a}^{m} i r(i), \ m \to \infty. \tag{A. 5}$$

Then, denoting $c \mathrel{\hat{=}} s^{-2} H(2H - 1)$ and applying Lemma A. 1 to $-b$ rvf $r(x)$, we get

$$\sum_{i=a}^{m} r(i) \sim c \int_a^m L(x) x^{-b} dx \sim \frac{cL(m) m^{-b+1}}{1 - b}, \quad m \to \infty \tag{A. 6}$$

where we used the Karamata equation (Proposition 1. 5. 8 in [2]),

$$\int_a^m L(x) x^u dx \sim L(m) \int_a^m x^u dx, \quad m \to \infty.$$

Similarly,

$$\sum_{i=a}^{m} ir(i) \sim \frac{cL(m)m^{-b+2}}{2-b}, \quad m \to \infty. \qquad \text{(A. 7)}$$

The equations (A. 4) - (A. 7) gives

$$\sum_{i=a}^{m} (m-i)r(i) \sim \frac{1}{2} s^{-2} L(m)m^{-b+2}, \quad m \to \infty. \qquad \text{(A. 8)}$$

Finally, from (A. 2) and (A. 8), we get

$$V_m \sim s^2 m^{-1} + L(m)m^{-b} \sim L(m)m^{-b}. \quad \therefore$$

APPENDIX B: PROOF OF THEOREM 3

Proof of Theorem 3. Consider a process $Y(l)$. Its sources have the deterministic active-period length l. A number of source epochs equaled t is denoted in 3. 1 by $x_{t,l}$. They are i. i. d. and Poissonian with parameter l_l.

For Y, let $\Omega(i,t,l)$ be the number of cells generated at t by the sources arrived at i with active-period length l, $t-l+1 \le i \le t$. In other words, $\Omega(i,t,l)$ is the number of cells generated at t by the sources arrived at i in process $Y(l)$. By $q(u,i,t-i+1,l)$ denote the contribution (i. e. the number of cells generated at t) to $\Omega(i,t,l)$ from the source u, $1 \le u \le x_{i,l}$ arrived at i with active-period length l,

$$\Omega(i,t,l) = q(1,i,t-i+1,l)+\ldots+q(x_{i,l},i,t-i+1,l). \qquad \text{(B. 1)}$$

(We underline that the source numbering by s in process Y (see 3. 1) has no relation with the source labeling by u which we use now.) The random variables $q(u,i,t-i+1,l)$ are independent for different triples (u,i,l).

Since

$$\mathrm{E}q(u,i,t-i+1,l) = m^{(l)},\ \mathrm{E}q(u,i,t_1-i+1,l)q(u,i,t_2-i+1,l) = B^{(l)}(|t_1-t_2|),$$

we have for

$$t_1 - l + 1 \le i \le t_1,\ t_2 - m + 1 \le j \le t_2, \qquad \text{(B. 2)}$$

that

$$\mathrm{E}q(u,i,t_1-i+1,l)q(v,j,t_2-j+1,m) = \begin{cases} B^{(l)}(|t_1-t_2|), \text{ if } (u,i,l)=(v,j,m), \\ m^{(l)}m^{(m)} \qquad \text{otherwise.} \end{cases} \qquad \text{(B. 3)}$$

The above equations will be used to express $\mathrm{E}\Omega(i,t,l)$ and $\mathrm{cov}\{\Omega(i,t,l),\Omega(j,t+k,m)\}$. We have

$$\mathrm{E}\Omega(i,t,l) = (\mathrm{E}x_{i,l})m^{(l)} = l \ \Pr\{t = l\} m^{(l)}, \qquad \text{(B. 4)}$$

and for the region (B. 2),

$$\mathrm{E}\Omega(i,t_1,l)\Omega(j,t_2,m) =$$

$$= \begin{cases} (l \ \Pr\{t = l\} m^{(l)})^2 + l \ \Pr\{t = l\} B^{(l)}(|t_1 - t_2|), \text{ if } (i,l)=(j,m) \\ l^2 \Pr\{t = l\}\Pr\{t = m\} m^{(l)} m^{(m)} \qquad \text{otherwise} \end{cases} \qquad \text{(B. 5)}$$

From (B. 4) and (B. 5), it is follows that

$$\mathrm{cov}\{\Omega(i,t,l),\Omega(j,t+k,m)\} = l \ \Pr\{t = l\} B^{(l)}(k) d_{ij} d_{lm} \qquad \text{(B. 6)}$$

for region (B. 2) with $t_1 = t, t_2 = t + k$.

Next and final step is to find the mean and the correlation coefficient of $Y = (\ldots, Y_{-1}, Y_0, Y_1, \ldots)$. Since

$$Y_t = \sum_{l=1}^{\infty} \sum_{i=t-l+1}^{t} \Omega(i,t,l), \qquad \text{(B. 7)}$$

we obtain (3. 3) using (B. 4).

Since $\Omega(i,t,l)$ are independent for different (i,l), we get from (B. 7) that

$$\operatorname{cov}\{Y_t, Y_{t+k}\} = \sum_{l=1}^{\infty} \sum_{i=t-l+1}^{t} \sum_{m=1}^{\infty} \sum_{j=t+k-m+1}^{t+k} \operatorname{cov}\{\Omega(i,t,l), \Omega(j,t+k,m)\}. \qquad \text{(B. 8)}$$

The equations (B. 6) and (B. 8) give (3. 4). ∴

APPENDIX C: PROOF OF THEOREM 4

Proof of Theorem 4 (2). In the proof, we shall use the expressions (3. 3) and (3. 4) for mean and autocovariance function of process Y. Here, the autocovariance function is denoted as $w(k) \hat{=} \operatorname{cov}\{Y_t, Y_{t+k}\}$.

First, we show that (3. 7) and (3. 8) are the necessary conditions for exact self-similarity of Y. When Y is ess, we have $\mathrm{E}Y_t < \infty$, $s^2 \equiv w(0) < \infty$, and

$$r(k) \hat{=} \frac{w(k)}{w(0)} = \frac{1}{2} d^2(k^{2-b}), \quad k \in I_1. \qquad \text{(C. 1)}$$

It means that $0 < s^2 < \infty$. Hence, using (3. 4), we have

$$0 < \sum_{l=1}^{\infty} \Pr\{t = l\} B^{(l)} \hat{=} a^{-1} < \infty.$$

Let us introduce a random variable t^* with

$$\Pr\{t^* = k\} = a \Pr\{t = k\} B^{(k)}, \quad k \in I_1. \qquad \text{(C. 2)}$$

The equations (3. 4) and (C. 2) give

$$w(k) = l\, a^{-1} \sum_{l=k+1}^{\infty} \Pr\{t^* \ge l\}, \quad k \in I_0. \qquad \text{(C. 3)}$$

It follows from (C. 1) and (C. 3) that

$$\Pr\{t^* = k\} = l^{-1} a d^2(w(k)) = l^{-1} a w(0) d^2(r(k)) \qquad \text{(C. 4)}$$

or

$$\Pr\{t^* = 1\} = \frac{a s^2}{2 l} (3^{2-b} - 2^{4-b} + 7), \qquad \text{(C. 5)}$$

$$\Pr\{t^* = k\} = \frac{a s^2}{2 l} d^4(k^{2-b}), \quad k \in I_2. \qquad \text{(C. 6)}$$

[The equation (C. 6) (regarding it holds for $k \in I_1$) was obtained by Cox [6] as a sufficient condition for exact self-similarity of Y in the case $m^{(l)} = B^{(l)} = 1$.]

Summing (C. 5)-(C. 6) over $1 \le k < \infty$, we get

$$a s^2 = 2 l \, [3^{2-b} - 2^{4-b} + 7 + \sum_{k=2}^{\infty} d^4(k^{2-b})]^{-1}. \qquad \text{(C. 7)}$$

The sum over $2 \le k < \infty$ can be calculated,

$$\sum_{k=2}^{\infty} d^4(k^{2-b}) = 3(2^{2-b} - 3^{1-b} - 1), \quad 0 < b < 1, \qquad \text{(C. 8)}$$

giving

$$as^2 = \frac{l}{2-2^{1-b}}. \quad \text{(C. 9)}$$

[Note that if (C. 5)-(C. 6) are multiplied by k and $k\Pr\{t^* = k\}$ is summed over $1 \le k < \infty$, we get the equation

$$3^{2-b} - 2^{4-b} + 7 + \sum_{k=2}^{\infty} kd^4(k^{2-b}) = 2 \quad \text{(C. 10)}$$

which is an identity when $0 < b < 1$.]

The condition (3. 7) follows from (C. 2), (C. 5), (C. 6), (C. 7), and (C. 9). The condition (3. 8) follows from $\mathrm{E}Y_t < \infty$, $0 < a < \infty$, (3. 7), and (3. 3).

Second, we show that (3. 7)-(3. 8) are the sufficient conditions for exact self-similarity of Y. The sufficiency easy follows from inversability of relations between $w(k)$ and $\Pr\{t^* = k\}$ and between $w(k)$ and $r(k)$ (see (C. 1), (C. 3), and (C. 4)) and from fulfilment of $s^2 < \infty$ (because of (C. 9)) and $\mathrm{E}Y_t < \infty$ (because of (C. 6) and (3. 3)). ∴

Proof of Theoren 4 (3). We use the same notations as above.

With (3. 4) and

$$w(k) = l\, B(0) r(k) \mathrm{E}t, \quad k \in I_0, \quad \text{(C. 11)}$$

we get

$$\Pr\{t = k\} = \left[\frac{r(k+1)}{B(k+1)} - 2\frac{r(k)}{B(k)} + \frac{r(k-1)}{B(k-1)}\right] B(0)\mathrm{E}t, \quad k \in I_1. \quad \text{(C. 12)}$$

If Y is es-s with $H = 1 - \frac{b}{2}$, $0 < b < 1$, it has $r(k) = \frac{1}{2} d^2(k^{2-b})$, $k \in I_1$, $r(0) = 1$ and (C. 12) gives

$$\Pr\{t = 1\} = \left[\frac{3^{2-b} - 2^{3-b} + 1}{2B(2)} + \frac{2 - 2^{2-b}}{B(1)} + \frac{1}{B(0)}\right] B(0)\mathrm{E}t, \quad \text{(C. 13)}$$

$$\Pr\{t = k\} = \frac{B(0)\mathrm{E}t}{2} d^2\left(\frac{d^2(k^{2-b})}{B(k)}\right), \quad k \in I_2. \quad \text{(C. 14)}$$

Summing (C. 13)-(C. 14) over $1 \le k < \infty$, expressing $\mathrm{E}t$,and useing the equation

$$\sum_{k=2}^{\infty} d^2\left(\frac{d^2(k^{2-b})}{B(k)}\right) = \frac{d^2(k^{2-b})|_{k=1}}{B(1)} - \frac{d^2(k^{2-b})|_{k=2}}{B(2)}, \quad 0 < b < 1,$$

we get (3. 13) .

Now, (C. 13),(C. 14) ,and (3. 13) give (3. 11)-(3. 12) and we proved that (3. 11)-(3. 12) are necessary for exact self-similarity of Y.

[Note that if we multiply (C. 13)-(C. 14) by k and sum $k\Pr\{t = k\}$ over $1 \le k < \infty$, we get the equation which is an identity for $0 < b < 1$ since

$$\sum_{k=2}^{\infty} kd^2\left(\frac{d^2(k^{2-b})}{B(k)}\right) = \frac{2^{3-b} - 4}{B(1)} - \frac{3^{2-b} - 2^{3-b} + 1}{B(2)}, \quad 0 < b < 1.]$$

The sufficiency of (3. 11)-(3. 12) can be easy proved with inversability argument as above and with check that $\mathrm{E}Y_t < \infty$, $s^2 < \infty$ when (3. 11)-(3. 12) hold. ∴

Proof of Theorem 4 (4). It is a corollary of Theorem 4 (3). ∴

Proof of Theorem 4 (1). It evidently follows from the definition of es-s process and from (3. 3) and (3. 4). Note that the condition $s^2 < \infty$ follows from (3. 5) and the condition $\mathrm{E}Y_t < \infty$ is guaranteed by (3. 6). ∴

APPENDIX D: PROOF OF THEOREM 5

Proof of Theorem 5. The equation (3. 4) gives

$$w(l) \hat{=} \mathrm{cov}\{Y_t, Y_{t+l}\} = l \sum_{k=l+1}^{\infty} \sum_{n=k}^{\infty} \Pr\{t = n\} B^{(n)}(l). \quad \text{(D. 1)}$$

Denote $f(l) \hat{=} \Pr\{t = l\} B^{(l)}(l)$, $l \in I_1$. According to (3. 17), the function $f(l)$ can be extended on a continuous neighbourhood of infinity, $[a, \infty)$, $a \geq 1$ as a $-(b+2)$ rvf, i. e. $f(x) \sim L(x) x^{-(b+2)}$, where svf $L(x)$ is locally bounded in $[a, \infty)$. Here and below without loss of generality, we assume that a is an integer.

The equation similar to (A. 3) holds,

$$\sum_{n=k}^{\infty} f(n) \sim \int_k^{\infty} L(x) x^{-(b+2)} dx, \quad k \to \infty. \quad \text{(D. 2)}$$

To prove (D. 2), we use the relation $a_i \sim b_i$, $i \to \infty$ where $a_i \hat{=} f(i)$, $b_i \hat{=} \int_i^{i+1} f(x) dx$. This relation is proved in the proof of Lemma A. 1.

Since $\sum a_i$ and also $\sum b_i$ (where both sums are over $a \leq i < \infty$) converge, we have $\left|\sum b_i - \sum a_i\right| \leq \sum b_i |1 - (a_i / b_i)|$. If $a_i \sim b_i$, there exists $e(a)$ such that $\left|\sum b_i - \sum a_i\right| < e(a) \sum b_i \to 0, \quad a \to \infty$, and we have proved (D. 2).

According to direct Karamata's Theorem (see Theorem 1. 5. 11 in [2]), we get from (3. 17) and (D. 2) that

$$\sum_{n=k}^{\infty} \Pr\{t = n\} B^{(n)}(k) \sim \int_k^{\infty} L(x) x^{-(b+2)} dx \sim \frac{L(k) k^{-(b+1)}}{b+1}, \quad k \to \infty. \quad \text{(D. 3)}$$

By the same arguments, we get

$$\sum_{k=l+1}^{\infty} L(k) k^{-(b+1)} \sim \frac{L(l) l^{-b}}{b}, \quad l \to \infty. \quad \text{(D. 4)}$$

The equations (D. 1), (D. 3), and (D. 4) give

$$w(l) \sim \frac{l L(l) l^{-b}}{b(b+1)}, \quad l \to \infty. \quad \text{(D. 5)}$$

Finally, (D. 5) together with Theorem 2 prove Theorem 5. ∴

APPENDIX E: DEFINITIONS OF REGULAR VARIATION

Definition. A measurable function $f(x) > 0$ satisfying

$$f(ux) / f(x) \to u^r, \quad x \to \infty$$

for each positive u, is called the *index* r *regularly varying function (rvf).*

If $r = 0$, then rvf $f(x)$ is called the *slowly varying function (svf).*

If $f(x)$ is an index r rvf then $f(x) = L(x)x^r$ where $L(x)$ is a svf.

Definition. A sequence $f(0), f(1), \ldots$ of positive nubmers is called the *index* r *regularly varying sequence (rvs) with continuous variable* u if

$$f([un]) / f(n) \to u^r, \qquad \text{integer } n \to \infty \tag{E. 1}$$

for each positive u where $[x]$ is the integer part of x.

If $r = 0$, then a rvs $f(0), f(1), \ldots$ with continuous variable u is called the *slowly varying sequence (svs) with continuous variable*.

Definition. A sequence $f(0), f(1), \ldots$ of positive numbers is called the *index* r *rvs with integer variable* m if

$$f(mn) / f(n) \to m^r, \qquad \text{integer } n \to \infty \tag{E. 2}$$

for each positive integer m.

If $r = 0$, then a rvs $f(0), f(1), \ldots$ with integer variable m is called the *svs with integer variable.*