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ON THE PAPER BY R.R. NIGMATULLIN "FRACTIONAL INTEGRAL AND ITS PHYSICAL INTERPRETATION"

The goal of establishing a clear relationship between fractial geometry and fractional calculus has been long sought by the scientific community. A number of intuitive or heuristic suggestions in this direction have been made in the last decade, Le Méhauté [1] having come the closest to a rigorous treatment of the problem. In this line of thinking, the paper by Nigmatullin [2] is an attempt to interpret fractional integration in terms of the fractal Cantor set. The author considers the evolution of the state of a physical system through its input–output relation

$$J(t) = \int_0^t K(t,\tau) f(\tau)\, dt$$

and suggests that fractional integration of the order ν can be interpreted by the above equation, with the impulse response (memory function) $K(t,\tau)$ providing for loss of some states of the system. This is achieved, according to the author, with the memory function generated by the Cantor set of the fractial dimension ν. This set is built iteratively by deleting, at the first step, the middle symmetrical part of length $2(1-\xi)t$ $\left(\xi = 2^{-\nu}, 0 < \xi \le \frac{1}{2}\right)$ of the interval $[0,t]$ and at each following step repeating a similar operation on all remaining intervals. The "intensity" of the memory function is kept constant by a scaling factor inverse to the summary area of the strips.

To this aim the author considers rectangular waveforms of a unit magnitude (2.8)[1)]

$$\eta(t_1 < \tau < t_2) = \begin{cases} 1 & \text{if } \tau \in [t_1, t_2], \\ 0 & \text{otherwise.} \end{cases}$$

The linear combinations of such functions reduced consequently by a scaling factor serve as the memory function in the above integral. By this design, the author believs apparently to have reduced $K(t,\tau)$ to $K(t-\tau)$, which should justify the use of the convolution theorem, therefore the Laplace transform of $J(t)$ is taken equal to the product of $\hat{K}(p)$ and $F(p)$, the corresponding Laplace transforms of $K(t)$ and $f(t)$. This way relation (2.12) is obtained, crucial for the results presented in the paper.

1) Formula (8) of the paper [2] and so on.

However, by the design used by the author, both the scaling factor and the width of the intervals $[t_1, t_2]$ become dependent on t, as can be seen from (2.10):

$$t_{m+1}^{(N+1)} = t_m^{(N)}, \quad t_{m+2}^{(N+1)} = t_m + \xi^{N+1}t,$$
$$t_{m+3}^{(N+1)} = t_{m+1}^{(N)} - \xi^{N+1}t, \quad t_{m+1}^{(N)}$$

(here N is the index of the step).

Obviously, the Cantor set as a memory function allows for no convolution. In fact,

$$J(t) = \frac{1}{(2\xi)^{2N}t} \sum_{m=1}^{2^{2N+1}} \left[\int_0^{t_{2m}} f(\tau)\,d\tau - \int_0^{t_{2m-1}} f(\tau)\,d\tau \right]$$

and

$$H(p) = \frac{1}{(2\xi)^{2N}} \sum_{m=1}^{2^{N+1}} \left[F\left(\frac{p}{\beta_{2m}}\right) - F\left(\frac{p}{\beta_{2m-1}}\right) \right],$$

where the constants $\beta_m = t_m/t$.

If we consider now for clarity just the first step of the divisions in the Cantor set ($N = 1$), (2.11) should read

$$J(t) = \frac{1}{2\xi t} \left[\int_0^t f(\tau)\,d\tau - \int_0^{(1-\xi)t} f(\tau)\,d\tau + \int_0^{\xi t} f(\tau)\,d\tau \right], \tag{1}$$

an expression which may very well not have a Laplace transform even for very common functions $f(t)$. But even if we drop the factor $1/t$ and introduce the functions

$$h(t): \quad J(t) = \frac{1}{t}h(t),$$
$$g(t): \quad = \int_0^t f(\tau)\,d\tau,$$

we get, for $N = 1$,

$$\begin{aligned} H(p) &= \frac{1}{2\xi}\left[G(p) + \frac{1}{\xi}G\left(\frac{p}{\xi}\right) - \frac{1}{(1-\xi)}G\left(\frac{p}{1-\xi}\right) \right] \\ &= \frac{1}{2\xi p}\left[F(p) + F\left(\frac{p}{\xi}\right) - F\left(\frac{p}{\xi}\right) \right], \quad \text{Re}p > 0, \end{aligned} \tag{2}$$

rather than

$$\hat{J}(p) = \frac{1 - \exp\{-p\xi t\}}{2\xi p\tau} [1 + \exp\{-pt(1-\xi)\}] F(p), \tag{3}$$

as obtained from (2.12) in the text. In a way of verifying, one can see that setting $\xi = 1/2$ in (2) leads to a correct Laplace transform for the conventional integration of the order $\nu = 1$, whereas the same substitution into (3) does not. A similar mistake in the introductory Section 1 of the paper has lead to the bizarre formulas (2.6) and (2.7).

Futhermore, if we set as an example $f(t) = 1$ for $t > 0$, the integration with the Cantor set as a memory function will result in the conventional integration ($\nu = 1$) for any ξ and any number of steps N, including $N \to \infty$. This can be easily seen either in the time domain from (1) or using the correct Laplace transform (2).

The aforesaid shows clearly, in the opinion of this writer, the erroneousness of the derivations in the paper and of the interpretation of the fractal Cantor set as a realization of the fractional integral.

But what can one say about the underlying physical idea? May it be that this operation approximates, in a sense, the fractional integration?

In order to answer this question, consider another example of

$$F(p) = \frac{1}{s+a}, \qquad a > 0, \qquad \mathrm{Re}\, p > -a.$$

Then

$$H(p) = \frac{1}{(2\xi)^{2N}} \sum_{m=1}^{2^{N+1}} \left[\frac{\beta_{2m}}{p + \alpha\beta_{2m}} - \frac{\beta_{2m-1}}{p + \alpha\beta_{2m-1}} \right]$$

i.e., the poles of the output Laplace transform make a sequence starting at the poles of the Laplace transform $H(p)$ and tending to 0. It can be shown, however (Rutman, 1990), that in an approximation of fractional integral by an infinite product, the poles form a sequence tending to $-\infty$, which describes an entirely different physical entity.

References

[1] *Le Méhaute A.* Les géometries fractales. Ed.Hermes. Paris: 1990.

[2] *Nigmatullin R.R.* Fractional integral and its physical interpretation // Theor.and Math.Phys. V. 90. P. 242–251; Russian original: TMF. 1992. V. 90. № 3. P. 354–368.

[3] *Rutman R.* On numerical integration and differention of fractional order: a systems theory approach. Theory and practice of geophysical data inversion (ed. A.Vogel et al), Vieweg, Braunschweig/Wiesbaden, 1992.

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