Recurrence and transience of branching random walks are dynamically stable

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Abstract

Consider a sequence of i.i.d. random variables X_n where each random variable is refreshed independently according to a Poisson clock. At any fixed time t the law of the sequence is the same as for the sequence at time 0 but at random times almost sure properties of the sequence may be violated. If there are such *exceptional times* we say that the property is *dynamically sensitive*, otherwise we call it *dynamically stable*. In this note we consider branching random walks on Cayley graphs and prove that recurrence and transience are dynamically stable. Our proof combines techniques from the theory of branching random walks with those of dynamical percolation.

Keywords: branching random walk, recurrence and transience, dynamical sensitivity
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1 Introduction

In Benjamini et al. [1] several properties of i.i.d. sequences are studied in the dynamical point of view. In particular, it is proven that transience of the simple random walk on the lattice $\mathbb{Z}^d$ is dynamically stable for $d \geq 5$, and dynamically sensitive for $d = 3, 4$. While recurrence is dynamically stable for $d = 1$, see [1], it is dynamically sensitive in dimension $d = 2$, compare with Hoffman [7]. Khoshnevisan studied in [9] and [10] other properties of dynamical random walks. We also refer to a recent survey [12] on dynamical percolation. In this note we define dynamical branching random walks on Cayley graphs and prove that recurrence and transience are dynamically stable, compare with Theorem 2.1. This result carries over to branching random walks on infinite connected graphs, compare with Remark 2.2.

1.1 Random Walks

Let us first collect the necessary notations for random walks on groups; for more details we refer to [14]. Let G be a finitely generated group with group identity e, the group operations are written multiplicatively (unless G is abelian). Let q be a probability measure on a finite generating set of G. The random walk on G with law q is the Markov chain with state space G and transition probabilities $p(x,y) = q(x^{-1}y)$ for $x, y \in G$. Equivalently, the process can be described on the product space $(G,q)^{\mathbb{N}}$: the nth projections X_n of $G^{\mathbb{N}}$ onto G constitute a sequence of independent G-valued random variables with common distribution q. Hence the random walk starting in $x \in G$ can be described as

$$S_n = xX_1 \cdots X_n, \quad n \geq 0.$$

If not mentioned otherwise the random walk starts in the group identity e. Let P denote the transition kernel of the random walk and $p^{(n)}(x,y) = \mathbb{P}(S_n = y | S_0 = x)$ be the probability to go from x to y in n steps. We will assume the random walk to be irreducible, i.e., for all x, y there exists some k such that $p^{(k)}(x,y) > 0$. Denote $G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n$ the corresponding generating functions. The inverse of the convergence radius of $G(x,y|z)$ is denoted by $\rho(P)$. There are two well known properties of $\rho(P)$:

$$\rho(P) = \limsup_{n\to\infty} \left(p^{(n)}(x,y)\right)^{1/n} = \sup_{|F|<\infty} \rho(P_F), \tag{1}$$

where $\rho(P_F)$ is the largest eigenvalue of the matrix $(P_F(x,y))_{x,y\in F}$ defined by $P_F(x,y) = P(x,y)$ for $x, y \in F$.

1.2 Branching Random Walks

We use the interpretation of tree-indexed random walks, compare with [2], to define the branching random walk (BRW). Let $\mathcal{T}$ be tree with root $\mathbf{r}$. For a vertex v of $\mathcal{T}$ let $|v|$ be the (graph) distance from v to the root $\mathbf{r}$. We label the edges of $\mathcal{T}$ with i.i.d. random variables with distribution q. The random variable X_v is the label of the edge (v^-, v) where v^- is the unique predecessor of v, i.e., $|v^-| = |v| - 1$. Define $S_v = e \cdot \prod_{i=1}^{|v|} X_{v_i}$ where $\langle v_0 = \mathbf{r}, v_1, \ldots, v_{|v|} = v\rangle$ is the unique geodesic from $\mathbf{r}$ to v. Note that the process is described on the product space $(G,q)^{\mathcal{T}}$.

A tree-indexed random walk becomes a BRW if the underlying tree is a realization of a Galton–Watson process with offspring distribution $\mu = (\mu_0, \mu_1, \ldots)$ and mean $m = \sum_k k\mu_k$. For ease of presentation we will assume that the Galton–Watson process survives almost sure, i.e., $\mu_0 = 0$, and that $m > 1$ in order to exclude the trivial case $\mu_1 = 1$. We say the BRW is *recurrent* if $\mathbb{P}(S_v = 0 \text{ for infinitley many } v) = 1$ and *transient* if $\mathbb{P}(S_v = 0 \text{ for infinitley many } v) = 0$. Here $\mathbb{P}$ does correspond to the product measure of the Galton–Watson process and the tree-indexed random walk: we pick a realization $\mathcal{T}(\omega)$ of the Galton–Watson process according to μ and define the BRW as the tree-indexed random walk on $\mathcal{T}(\omega)$. Alternatively we could say that the BRW is recurrent if for a.a. realization $\mathcal{T}(\omega)$ the tree-indexed random walk is recurrent, i.e., $\mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{|v|=n} \mathbf{1}\{S_v = e\} = \infty\right) = 1$, where $\mathbb{P}$ corresponds to $(G,q)^{\mathcal{T}(\omega)}$.

We have the following classification due to [4]:

Theorem 1.1. *The BRW is transient if and only if $m \leq 1/\rho(P)$.*

Remark 1.1. There is the following equivalent description of BRW. At time 0 we start the process with one particle in e. At time 1 this particle splits up according to some offspring distribution μ. Then these offspring particles move (still at time 1) independently according to P. The BRW is now defined inductively: at each time each particle splits up independently of the others according to μ and the *new* particles move then independently according to P.

2 Dynamical BRW

Let us introduce the dynamical process. Fix a tree $\mathcal{T}$. For each $v \in \mathcal{T}$, let $\{X_v(t)\}_{t\geq 0}$ be an independent process that updates its value by an independent sample of q with rate 1. Formally, consider i.i.d. random variables $\{X_v^{(j)} : v \in \mathcal{T},\ j \in \mathbb{N}\}$ with law q, and an independent Poisson process $\{\psi_v^{(j)}\}_{j\geq 0}$ of rate 1 for each $v \in \mathcal{T}$. Define

$$X_v(t) := X_v^{(j)} \text{ for } \psi_v^{(j-1)} \leq t < \psi_v^{(j)}, \tag{2}$$

where $\psi_v^{(0)} = 0$ for every n. The distribution of $(X_v(t))_{v\in\mathcal{T}}$ is $q^{\mathcal{T}}$ for every $t \geq 0$. Denote $\mathbf{P}$ the probability measure on the underlying probability space on which the dynamical BRW process is defined. In the following $\mathbf{P}$, $\mathbf{E}$ will always correspond to the dynamical version while $\mathbb{P}$, $\mathbb{E}$ describe the non-dynamical process.

Due to Theorem 1.1 we have with Fubini's Theorem

$$\mathbf{P}\left(\sum_{n=1}^{\infty}\sum_{|v|=n} \mathbf{1}\{S_v(t) = e\} = \infty \text{ for Lebesgue-a.e. } t\right) = 0 \tag{3}$$

if $m \leq 1/\rho$ and equals 1 if $m > 1/\rho(P)$. The result of this note is that there are no exceptional times for transience and recurrence of BRW:

Theorem 2.1. *We consider a BRW on a Cayley graph G with law q and offspring distribution μ (whose support excludes 0) and mean $m > 1$. Then we have:*

- *if $m \leq 1/\rho(P)$ then*

$$\boldsymbol{P}\left(\sum_{n=1}^{\infty}\sum_{|v|=n} \boldsymbol{1}\{S_v(t) = e\} = \infty \textit{ for all } t\right) = 0$$

- *if $m > 1/\rho(P)$ then*

$$\boldsymbol{P}\left(\sum_{n=1}^{\infty}\sum_{|v|=n} \boldsymbol{1}\{S_v(t) = e\} = \infty \textit{ for all } t\right) = 1$$

Remark 2.1. The statement of Theorem 2.1 holds true if we use a dynamical Galton–Watson process to define the dynamical BRW. Observe hereby that supercriticality of a Galton–Watson process is dynamically stable and that Equation (4) can be adapted without changing the fact that $\sum \mathbf{P}(Z_n > 0) < \infty$.

Remark 2.2. Theorem 2.1 can be generalized to branching random walks on graphs and even Galton–Watson processes with an infinite number of types. Observe hereby that if the Galton–Watson process has an infinite number of types we can speak of local and global survival. Local survival means that every type survives with positive probability and global survival that the process on its own survives with positive probability, compare with [5]. Theorem 2.1 generalizes to: there are no exceptional times for local survival for Galton–Watson processes with an infinite number of types. The treatment of global survival is in general more difficult and even more subtle since *critical* processes may survive or die out. Therefore the study of exceptional times for global survival/extinction is one of the next steps to go.

Remark 2.3. Let us consider a transient random walk $S_n = \sum_{i=1}^n X_i$ on $\mathbb{Z}$ (or $\mathbb{R}$) with $\mathbb{E}[X_i] > 0$. We assume that there exists a rate function $I(\cdot)$ satisfying

$$-I(a) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an) \text{ for } a \leq \mathbb{E}[X_i].$$

Denote by m_n the minimal position of a particle at time n; $m_n = \min_{|v|=n} S_v$. There is the classical result that $\lim_{n\to\infty} \frac{m_n}{n} = \inf\{s : I(s) \leq \log m\}$. Combining the proof of Theorem 18.3 in [11] with the ideas of the proof of Theorem 2.1 one can see that there are no exceptional times for the (linear) speed, i.e., $\lim_{n\to\infty} \frac{m_n(t)}{n} = \inf\{s : I(s) \leq \log m\}$ for all t. Furthermore, in the critical case $m = 1/\rho(P)$ we have that $m_n/n \to 0$ but $m_n \to \infty$ as $n \to \infty$. The *second order* behaviour is more subtle: while for a wide range of BRW $m_n/\log n$ converges in probability it does in general not converge almost sure. We refer to [8] for more details and references on recent results. In this respect the study of exceptional times for the second order behaviour is of interest.

3 Proof of Theorem 2.1

The proof relies on the fact that recurrence of BRW is equivalent to the existence of *(recurrent) seeds* that were introduced in [3]. A seed is a finite subgraph such that the BRW restricted to this subgraph may *explode* (or survive) with positive probability. The particles leaving the seed then eventually *fill* the whole graph. To make this more precise let us introduce the following notations.

We denote $B_N = \{x \in G : d(e,x) \leq N\}$ the ball of radius N around e in standard word metric $d(\cdot,\cdot)$. Let us consider a truncated version S_n^N of the random walk S_n where we kill the random walk and bury it in $\dagger$ if it leaves the ball B_N. More formally, let $S_n^N = S_n$ if $S_i \in B_N$ for all $i \leq n$ and $\dagger$ otherwise. This random walk induces a branching random walk on $B_N \cup \{\dagger\}$ in the following way: $S_v^N = S_v$ if $S_w \in B_N$ for all vertices w on the geodesic from $\mathbf{r}$ to v and $S_v^N = \dagger$ otherwise.

Lemma 3.1. *If BRW on G is recurrent then a.s. there exists a (random) K such that*

$$\sum_n \sum_{|v|=n} \mathbf{1}\{S_v^K = e\} = \infty$$

Proof. We make use of the *particle* interpretation of the BRW, compare with Remark 1.1, and consider a sequence of embedded branching processes. As in [5] we construct an embedded Galton–Watson process counting the number of particles in the origin. The first generation of this process is formed by those particles that are the first in their ancestry line (of the BRW) to return to e. The process is defined inductively: the i-th generation consists of those particles that are the i-th particle in their ancestry line to return to e. Denote by $\psi_i^{(1)}$ the size of the i-th generation. Observe that $\psi_i^{(1)} \in \mathbb{N} \cup \{\infty\}$ is a Galton–Watson process with mean $\mathbb{E}\psi_1^{(1)} > 1$ since the BRW is recurrent. We define a truncated version of the latter process by counting only those particles whose ancestors were all in the ball of radius N around e. Let $\psi_i^{(N,1)}$ be the size of the truncated i-th generation and let us choose N such that $\mathbb{E}\psi_1^{(N,1)} > 1$. Hence $\psi_i^{(N,1)}$ is a supercritical Galton–Watson process that survives with positive probability q. If this first process dies out, we define a second process $\psi_i^{(N,2)}$ analogously to the first one but where e is replaced by some position that is occupied by some particles in B_{N+1} at the time when the first process dies out, i.e., the time when eventually all particles have left B_N. Again, the process $\psi_i^{(N,2)}$ survives with positive probability q. Inductively we obtain a sequence of independent supercritical Galton–Watson processes with extinction probabilities $1-q<1$. Hence, there exists a.s. some j^* such that the process $\psi_i^{(N,j^*)}$ survives. Letting $K = N + j^*$ we have that $\sum_{|v|=n} \mathbf{1}\{S_v^K = e\} > 0$ for infinitely many n. □

Proof of Theorem 2.1
Transience is dynamically stable: It is convenient to define an auxiliary random variable τ, which is exponentially distributed with mean 1 and independent of $(X_v(t))_{v\in\mathcal{T},t\geq 0}$, compare with Section 3 in [1]. Let $N \in \mathbb{N}$ and define

$$Z_n^N := \int_0^\tau \sum_{|v|=n} \mathbf{1}\{S_v^N(t) = e\} dt.$$

For ease of presentation we omit the superscript N and just write Z_n for Z_n^N.

By Fubini's Theorem we have $\mathbf{E}[Z_n] = m^n p_N^{(n)}(e,e)$. Observe, that for every finite set F we have that $\rho(P) > \rho(P_F)$, compare with Chapter 2 in [13]. Hence $\mathbf{E}[Z_n] \leq q_N^n$ for some $q_N < 1$. We follow the line of proof of Lemma 5.6 in [1]. In what follows we only consider those $n \in \mathbb{N}$ such that $\mathbb{P}(S_n = e) > 0$. We have for $n \geq 1$

$$\mathbf{P}(Z_n > 0) = \frac{\mathbf{E}[Z_n]}{\mathbf{E}[Z_n | Z_n > 0]}.$$

Let $\sigma := \inf\{t \geq 0 : S_v(t) = e \text{ for some } v \in T_n\}$. Conditioned on the event $\{Z_n > 0\}$ we have $\sigma \in [0,\tau)$. Let $|v| = n$ be such that $S_v(\sigma) = e$ and $\langle \mathbf{r}, v_1, v_2, \ldots, v_n = v\rangle$ the geodesic from $\mathbf{r}$ to

v. By the strong Markov property and the *memoryless* property of τ, we have

$$\mathbf{P}[\tau > \sigma + 1/n, X_{v_k} \text{ does not change its value during } t \in [\sigma, \sigma + 1/n] | Z_n > 0] = \left(\frac{1}{e}\right)^{1/n} \frac{1}{e}. \quad (4)$$

If the above event occurs, then $Z_n \geq 1/n$. Hence

$$\mathbf{P}[Z_n \geq 1/n | Z_n > 0] \geq 1/e^2, \text{ and } \mathbf{E}[Z_n | Z_n > 0] \geq \frac{1}{ne^2}.$$

Eventually,

$$\mathbf{P}(Z_n > 0) \leq e^2 n q_N^n \quad \forall n$$

and hence $\sum_n \mathbf{P}(Z_n > 0) < \infty$. By the lemma of Borel–Cantelli for all N there are no times t such that $\sum_{|v|=n} \mathbf{1}\{S_v^N(t) = e\} > 0$ for infinitely many n and therefore

$$\mathbf{P}\big(\exists N \; \exists t : \sum_n \sum_{|v|=n} \mathbf{1}\{S_v^N(t) = e\} = \infty\big) = 0.$$

Dynamical stability of transience now follows with Lemma 3.1.
Recurrence is dynamically stable: Let us first consider the non-dynamical BRW. Since $m < 1/\rho(P)$ we have with equation (1) that there exists some $k \in \mathbb{N}$ such that $p^{(k)}(e,e) > m^{-k}$. It's rather standard, e.g. compare with proof of Theorem 18.3 in [11], to define the following embedded process $(\xi_n)_{n\geq 1}$ where we observe the process only at times ik and kill all particles that are not in e at these times. Then ξ_n describes the number of particles at e at time nk. Since $\mathbb{E}[\xi_1] = p^{(k)}(e,e)m^k > 1$ the process ξ_n is a supercritical Galton–Watson process. Hence, it can be written as $\xi_n = \sum_{i=1}^{\xi_{n-1}} Y_{n,i}$, where $Y_{n,i}$ are i.i.d. random variables with $\mathbb{E}[Y_{n,i}] = p^{(k)}(e,e)m^k > 1$. Now, let M be such that $\mathbb{E}[\min\{M, Y_{n,i}\}] > 1$ and let $X_{n,i} = \min\{M, Y_{n,i}\}$. The dynamical version of the branching random walk induces a dynamical process defined as

$$Z_n(t) = \sum_{i=1}^{Z_{n-1}(t)} X_{n,i}(t),$$

where $X_{n,i}(t)$ is the dynamical version of $X_{n,i}$. As before $\mathbb{P}$ denotes the probability measure for the non-dynamical process while $\mathbf{P}$ describes the dynamical version. Analogously to the study of the noncritical cases in [6] let for all indices i, n

$$\inf_{[a,b]} X_{n,i} = \inf_{t\in[a,b]} X_{n,i}(t),$$

and inductively

$$\inf_{[a,b]} Z_n = \sum_{i=1}^{\inf_{[a,b]} Z_{n-1}} \inf_{[a,b]} X_{n,i}.$$

Observe

$$\mathbf{P}(\inf_{[a,b]} X_{n,i} = k) \geq \mathbb{P}(X_{n,i} = k) \left(\frac{1}{e}\right)^{(b-a)M^{k+1}}$$

and hence for some $\varepsilon > 0$

$$\mathbf{E}[\inf_{[0,\varepsilon]} X_{n,i}] \geq m \left(\frac{1}{e}\right)^{\varepsilon} > 1.$$

Therefore, the Galton–Watson process defined by $\tilde{X}_{n,i} = \inf_{[0,\varepsilon]} X_{n,i}$ is supercritical and hence there are no exceptional times in the interval $[0,\varepsilon]$. Repeating the arguments for the intervals $[k\varepsilon,(k+1)\varepsilon]$ and using countable additivity concludes the proof for $m > 1$.

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