FRACALMO PRE-PRINT **ww.fracalmo.org**
Published in **Vietnam Journal of Mathematics** Vol. 32 (2004), SI pp. 53-64
Springer Verlag: ISSN 0866-7179

A fractional generalization of the Poisson processes

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Abstract

It is our intention to provide via fractional calculus a generalization of the pure and compound Poisson processes, which are known to play a fundamental role in renewal theory, without and with reward, respectively. We first recall the basic renewal theory including its fundamental concepts like waiting time between events, the survival probability, the counting function. If the waiting time is exponentially distributed we have a Poisson process, which is Markovian. However, other waiting time distributions are also relevant in applications, in particular such ones with a fat tail caused by a power law decay of its density. In this context we analyze a non-Markovian renewal process with a waiting time distribution described by the Mittag-Leffler function. This distribution, containing the exponential as particular case, is shown to play a fundamental role in the infinite thinning procedure of a generic renewal process governed by a power-asymptotic waiting time. We then consider the renewal theory with reward that implies a random walk subordinated to a renewal process.

MSC 2000: 26A33, 33E12, 44A10, 44A35, 60G50, 60G55, 60J05, 60K05.

Key words: Renewal theory, Poisson process, Erlang distribution, fractional derivatives, Mittag-Leffler function, thinning procedure, random walks.

1 Essentials of renewal theory

The concept of *renewal process* has been developed as a stochastic model for describing the class of counting processes for which the times between successive events are independent identically distributed (*iid*) non-negative random variables, obeying a given probability law. These times are referred to as waiting times or inter-arrival times. For more details see *e.g.* the classical treatises by Khintchine [12], Cox [2], Gnedenko & Kovalenko [6], Feller [5], and the recent book by Ross [19].

For a renewal process having waiting times $T_1, T_2, \ldots$, let

$$t_0 = 0\,, \quad t_k = \sum_{j=1}^{k} T_j\,, \quad k \geq 1\,. \tag{1.1}$$

That is $t_1 = T_1$ is the time of the first renewal, $t_2 = T_1 + T_2$ is the time of the second renewal and so on. In general t_k denotes the kth renewal.

The process is specified if we know the probability law for the waiting times. In this respect we introduce the *probability density function* (*pdf*) $\phi(t)$ and the (cumulative) distribution function $\Phi(t)$ so defined:

$$\phi(t) := \frac{d}{dt}\Phi(t)\,, \quad \Phi(t) := P\left(T \leq t\right) = \int_0^t \phi(t')\,dt'\,. \tag{1.2}$$

When the nonnegative random variable represents the lifetime of technical systems, it is common to refer to $\Phi(t)$ as to the *failure probability* and to

$$\Psi(t) := P\left(T > t\right) = \int_t^\infty \phi(t')\,dt' = 1 - \Phi(t)\,, \tag{1.3}$$

as to the *survival probability*, because $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in $(0, T]$. A relevant quantity is the *counting function* $N(t)$ defined as

$$N(t) := \max\left\{k | t_k \leq t,\ k = 0, 1, 2, \ldots\right\}\,, \tag{1.4}$$

that represents the effective number of events before or at instant t. In particular we have $\Psi(t) = P\left(N(t) = 0\right)$. Continuing in the general theory we set $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$, and in general

$$F_k(t) := P\left(t_k = T_1 + \ldots + T_k \leq t\right)\,,\ f_k(t) = \frac{d}{dt}F_k(t)\,,\ k \geq 1\,, \tag{1.5}$$

thus $F_k(t)$ represents the probability that the sum of the first k waiting times is less or equal t and $f_k(t)$ its density. Then, for any fixed $k \geq 1$ the normalization condition for $F_k(t)$ is fulfilled because

$$\lim_{t\to\infty} F_k(t) = P\left(t_k = T_1 + \ldots + T_k < \infty\right) = 1\,. \tag{1.6}$$

In fact, the sum of k random variables each of which is finite with probability 1 is finite with probability 1 itself. By setting for consistency $F_0(t) \equiv 1$ and $f_0(t) = \delta(t)$, the Dirac delta function[1], we also note that for $k \geq 0$ we have

$$P\left(N(t) = k\right) := P\left(t_k \leq t\,,\, t_{k+1} > t\right) = \int_0^t f_k(t')\,\Psi(t-t')\,dt'\,. \tag{1.7}$$

We now find it convenient to introduce the simplified $*$ notation for the Laplace convolution between two causal well-behaved (generalized) functions $f(t)$ and $g(t)$

$$\int_0^t f(t')\,g(t-t')\,dt' = (f\,*\,g)\,(t) = (g\,*\,f)\,(t) = \int_0^t f(t-t')\,g(t')\,dt'\,.$$

Being $f_k(t)$ the *pdf* of the sum of the k *iid* random variables $T_1, \ldots, T_k$ with *pdf* $\phi(t)$, we easily recognize that $f_k(t)$ turns out to be the k-fold convolution of $\phi(t)$ with itself,

$$f_k(t) = \left(\phi^{*k}\right)(t)\,, \tag{1.8}$$

so Eq. (1.7) simply reads:

$$P\left(N(t) = k\right) = \left(\phi^{*k}\,*\,\Psi\right)(t)\,. \tag{1.9}$$

Because of the presence of Laplace convolutions a renewal process is suited for the Laplace transform method. Throughout this paper we will denote by $\widetilde{f}(s)$ the Laplace transform of a sufficiently well-behaved (generalized) function $f(t)$ according to

$$\mathcal{L}\left\{f(t); s\right\} = \widetilde{f}(s) = \int_0^{+\infty} \mathrm{e}^{-st}\,f(t)\,dt\,, \quad s > s_0\,,$$

[1] We find it convenient to recall the *formal representation* of this generalized function in $\mathbf{R}^+$,

$$\delta(t) := \frac{t^{-1}}{\Gamma(0)}\,, \quad t \geq 0\,.$$

and for $\delta(t)$ consistently we will have $\widetilde{\delta}(s) \equiv 1$. Note that for our purposes we agree to take s real. We recognize that (1.9) reads in the Laplace domain

$$\mathcal{L}\{P\,(N(t)=k)\,;s\} = \left[\widetilde{\phi}(s)\right]^k \, \widetilde{\Psi}(s)\,, \tag{1.10}$$

where, using (1.3),

$$\widetilde{\Psi}(s) = \frac{1-\widetilde{\phi}(s)}{s}\,. \tag{1.11}$$

2 The Poisson process as a renewal process

The most celebrated renewal process is the Poisson process characterized by a waiting time *pdf* of exponential type,

$$\phi(t) = \lambda\,\mathrm{e}^{-\lambda t}\,, \quad \lambda > 0\,, \quad t \ge 0\,. \tag{2.1}$$

The process has *no memory*. Its moments turn out to be

$$\langle T\rangle = \frac{1}{\lambda}\,, \quad \langle T^2\rangle = \frac{1}{\lambda^2}\,, \quad \dots\,, \quad \langle T^n\rangle = \frac{1}{\lambda^n}\,, \quad \dots\,, \tag{2.2}$$

and the *survival probability* is

$$\Psi(t) := P\,(T > t) = \mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,. \tag{2.3}$$

We know that the probability that k events occur in the interval of length t is

$$P\,(N(t)=k) = \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,, \quad k = 0,1,2,\dots\,. \tag{2.4}$$

The probability distribution related to the sum of k *iid* exponential random variables is known to be the so-called *Erlang distribution* (of order k). The corresponding density (the *Erlang pdf*) is thus

$$f_k(t) = \lambda\,\frac{(\lambda t)^{k-1}}{(k-1)!}\,\mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,, \quad k = 1,2,\dots\,, \tag{2.5}$$

so that the Erlang distribution function of order k turns out to be

$$F_k(t) = \int_0^t f_k(t')\,dt' = 1 - \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t} = \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,. \tag{2.6}$$

In the limiting case $k=0$ we recover $f_0(t) = \delta(t)$, $F_0(t) \equiv 1$, $t \ge 0$.

The results (2.4)-(2.6) can easily obtained by using the technique of the Laplace transform sketched in the previous section noting that for the Poisson process we have:

$$\widetilde{\phi}(s) = \frac{\lambda}{\lambda + s}\,, \quad \widetilde{\Psi}(s) = \frac{1}{\lambda + s}\,, \tag{2.7}$$

and for the Erlang distribution:

$$\widetilde{f}_k(s) = [\widetilde{\phi}(s)]^k = \frac{\lambda^k}{(\lambda + s)^k}\,, \quad \widetilde{F}_k(s) = \frac{[\widetilde{\phi}(s)]^k}{s} = \frac{\lambda^k}{s(\lambda + s)^k}\,. \tag{2.8}$$

We also recall that the survival probability for the Poisson renewal process obeys the ordinary differential equation (of relaxation type)

$$\frac{d}{dt}\Psi(t) = -\lambda\Psi(t)\,, \quad t \geq 0\,; \quad \Psi(0^+) = 1\,. \tag{2.9}$$

3 The renewal process of Mittag-Leffler type

A "fractional" generalization of the Poisson renewal process is simply obtained by generalizing the differential equation (2.9) replacing there the first derivative with the integro-differential operator ${}_tD_*^\beta$ that is interpreted as the fractional derivative of order β in Caputo's sense, see Appendix. We write, taking for simplicity $\lambda = 1$,

$${}_tD_*^\beta\,\Psi(t) = -\Psi(t)\,, \quad t > 0\,, \quad 0 < \beta \leq 1\,; \quad \Psi(0^+) = 1\,. \tag{3.1}$$

We also allow the limiting case $\beta = 1$ where all the results of the previous section (with $\lambda = 1$) are expected to be recovered.

For our purpose we need to recall the Mittag-Leffler function as the natural "fractional" generalization of the exponential function, that characterizes the Poisson process. The Mittag-Leffler function of parameter β is defined in the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\,n + 1)}\,, \quad \beta > 0\,, \quad z \in \mathbf{C}\,. \tag{3.2}$$

It turns out to be an entire function of order β which reduces for $\beta = 1$ to $\exp(z)$. For detailed information on the Mittag-Leffler-type functions and their Laplace transforms the reader may consult *e.g.* [4, 8, 17].

The solution of Eq. (3.1) is known to be, see *e.g.* [1, 8, 13],

$$\Psi(t) = E_\beta(-t^\beta)\,, \quad t \geq 0\,, \quad 0 < \beta \leq 1\,, \tag{3.3}$$

so

$$\phi(t) := -\frac{d}{dt}\Psi(t) = -\frac{d}{dt}E_\beta(-t^\beta)\,, \quad t \ge 0\,, \quad 0 < \beta \le 1\,. \tag{3.4}$$

Then, the corresponding Laplace transforms read

$$\widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}\,, \quad \widetilde{\phi}(s) = \frac{1}{1+s^\beta}\,, \quad 0 < \beta \le 1\,. \tag{3.5}$$

Hereafter, we find it convenient to summarize the most relevant features of the functions $\Psi(t)$ and $\phi(t)$ when $0 < \beta < 1$. We begin to quote their series expansions for $t \to 0^+$ and asymptotics for $t \to \infty$,

$$\Psi(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+1)} \sim \frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta)}{t^\beta}\,, \tag{3.6}$$

and

$$\phi(t) = \frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+\beta)} \sim \frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta+1)}{t^{\beta+1}}\,. \tag{3.7}$$

In contrast to the Poissonian case $\beta = 1$, in the case $0 < \beta < 1$ for large t the functions $\Psi(t)$ and $\phi(t)$ no longer decay exponentially but algebraically. As a consequence of the power-law asymptotics the process turns be no longer Markovian but of long-memory type. However, we recognize that for $0 < \beta < 1$ both functions $\Psi(t)$, $\phi(t)$ keep the "completely monotonic" character of the Poissonian case. Complete monotonicity of the functions $\Psi(t)$ and $\phi(t)$ means

$$(-1)^n \frac{d^n}{dt^n}\,\Psi(t) \ge 0\,, \quad (-1)^n \frac{d^n}{dt^n}\,\phi(t) \ge 0\,, \quad n = 0, 1, 2, \ldots\,, \quad t \ge 0\,, \tag{3.8}$$

or equivalently, their representability as real Laplace transforms of non-negative generalized functions (or measures), see *e.g.* [8].

For the generalizations of Eqs (2.4) and (2.5)-(2.6), characteristic of the Poisson and Erlang distributions respectively, we must point out the Laplace transform pair

$$\mathcal{L}\{t^{\beta\, k}\, E_\beta^{(k)}(-t^\beta); s\} = \frac{k!\, s^{\beta-1}}{(1+s^\beta)^{k+1}}\,, \quad \beta > 0\,, \quad k = 0, 1, 2, \ldots\,, \tag{3.9}$$

with $E_\beta^{(k)}(z) := \dfrac{d^k}{dz^k}E_\beta(z)$, that can be deduced from the book by Podlubny, see (1.80) in [17]. Then, by using the Laplace transform pairs (3.5) and Eqs

(3.3), (3.4), (3.9) in Eqs (1.8) and (1.9), we have the *generalized Poisson distribution*,

$$P\left(N(t)=k\right)=\frac{t^{k\,\beta}}{k!}\,E_\beta^{(k)}(-t^\beta)\,,\quad k=0,1,2,\ldots \tag{3.10}$$

and the *generalized Erlang pdf* (of order $k\geq 1$),

$$f_k(t)=\beta\,\frac{t^{k\beta-1}}{(k-1)!}\,E_\beta^{(k)}(-t^\beta)\,. \tag{3.11}$$

The *generalized Erlang distribution function* turns out to be

$$F_k(t)=\int_0^t f_k(t')\,dt'=1-\sum_{n=0}^{k-1}\frac{t^{n\beta}}{n!}\,E_\beta^{(n)}(-t^\beta)=\sum_{n=k}^{\infty}\frac{t^{n\beta}}{n!}\,E_\beta^{(n)}(-t^\beta)\,. \tag{3.12}$$

4 The Mittag-Leffler distribution as limit for thinned renewal processes

The procedure of thinning (or rarefaction) for a generic renewal process (characterized by a generic random sequence of waiting times $\{T_k\}$) has been considered and investigated by Gnedenko and Kovalenko [6]. It means that for each positive index k a decision is made: the event is deleted with probability p or it is maintained with probability $q=1-p$, with $0<q<1$. For this thinned or rarefied renewal process we shall hereafter revisit and complement the results available in [6]. We begin to rescale the time variable t replacing it by t/r, with a parameter r on which we will dispose later. Denoting, like in (1.5), by $F_k(t)$ the probability distribution function of the sum of k waiting times and by $f_k(t)$ its density, we have recursively, in view of (1.8),

$$f_1(t)=\phi(t)\,,\quad f_k(t)=\int_0^t f_{k-1}(t-t')\,\phi(t')\,dt'=\left(\phi^{*k}\right)(t)\,,\quad k\geq 2\,. \tag{4.1}$$

Let us denote by $(T_{q,r}f)(t)$ the waiting time density in the thinned and rescaled process from one event to the next. Observing that after a maintained event the next one of the original process is kept with probability q but dropped in favour of the second next with probability $p\,q$ and, generally, $n-1$ events are dropped in favour of the n-th next with probability $p^{n-1}\,q$, we arrive at the formula

$$(T_{q,r}f)(t)=\sum_{n=1}^{\infty}q\,p^{n-1}\,f_n(t/r)/r\,. \tag{4.2}$$

Let $\widetilde{f}_n(s) = \int_0^\infty \mathrm{e}^{-st}\, f_n(t)\, dt$ be the Laplace transform of $f_n(t)$. Recalling $f_1(t) = \phi(t)$ we set $\widetilde{f}_1(s) = \widetilde{\phi}(s)$. Then $f_n(t/r)/r$ has the transform $\widetilde{f}_n(rs) = \left(\widetilde{\phi}(rs)\right)^n$, and we obtain (in view of $p = 1 - q$) the formula

$$(T_{q,r}\widetilde{\phi})(s) = \sum_{n=1}^{\infty} q\, p^{n-1} \left(\widetilde{\phi}(rs)\right)^n = \frac{q\,\widetilde{\phi}(rs)}{1-(1-q)\,\widetilde{\phi}(rs)}\,, \tag{4.3}$$

from which by Laplace inversion we can, in principle, construct the transformed process.

We now imagine stronger and stronger rarefaction (infinite thinning) by considering a scale of processes with the parameters $r = \delta$ and $q = \epsilon$ tending to zero under a scaling relation $\epsilon = \epsilon(\delta)$ yet to be specified. Gnedenko and Kovalenko have, among other things, shown that if the condition

$$\widetilde{\phi}(s) = 1 - a(s)\, s^\beta + o\left(a(s)\, s^\beta\right)\,, \quad \text{for} \quad s \to 0^+\,, \tag{4.4}$$

where $a(s)$ is a slowly varying function for $s \to 0$[2], is satisfied, then we have with $\epsilon = \epsilon(\delta) = a(\delta)\, \delta^\beta$ for every *fixed* $s > 0$ the limit relation

$$\widetilde{\phi}_0(s) := \lim_{\delta \to 0} \frac{\epsilon(\delta)\, \widetilde{\phi}(\delta s)}{1-(1-\epsilon)\, \widetilde{\phi}(\delta s)} = \frac{1}{1+s^\beta}\,, \qquad 0 < \beta \le 1\,. \tag{4.5}$$

This condition is met with $a(s) = \lambda\, M(1/s)$ if the waiting time T obeys a power law with index β, in the sense of *Master Lemma 2* by Gorenflo and Abdel-Rehim [7]. The function $M(y)$ is the same as in *Master Lemma 2*, so it varies slowly at infinity, whence $M(1/s)$ varies slowly at zero. The proof of (4.5) is by straightforward calculation. Observe the slow variation property of $a(s)$ and note that terms small of higher order become negligible in the limit. By the continuity theorem for Laplace transforms, see Feller [5], we now recognize $\phi_0(t)$ as the limiting density, which we identify, in view of (3.2)-(3.5),

$$\phi_0(t) = -\frac{d}{dt} E_\beta(-t^\beta)\,. \tag{4.6}$$

So the limiting waiting time density is the so-called Mittag-Leffler density, that in the special case $\beta = 1$ reduces to the well-known exponential density.

[2]**Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, *slowly varying at zero* if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, *slowly varying at infinity* if $b(cy)/a(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $(\log y)^\gamma$ with $\gamma \in \mathbf{R}$ and $\exp\,(\log y/\log\,\log y)$.

It should be noted that Gnedenko and Kovalenko in the sixties failed to recognize $\widetilde{\phi}_0(s)$ as Laplace transform of a Mittag-Leffler type function[3].

5 Renewal processes with reward: the fractional master equation and its solution

The renewal process can be accompanied by reward that means that at every renewal instant a space-like variable makes a random jump from its previous position to a new point in "space". "Space" is here meant in a very general sense. In the insurance business, e.g., the renewal points are instants where the company receives a payment or must give away money to some claim of a customer, so space is money. In such process occurring in time and in space, also referred to as *compound renewal process*, the probability distribution of jump widths is as relevant as that of waiting times.

Let us denote by X_n the jumps occurring at instants $t_n\,,\ n = 1, 2, 3, \ldots$. Let us assume that X_n are *iid* (real, not necessarily positive) random variables with probability density $w(x)$, independent of the *waiting time* density $\phi(t)$. In a physical context the X_ns represent the jumps of a diffusing particle (the walker), and the resulting random walk model is known as *continuous time random walk* (abbreviated as CTRW) in that the waiting time is assumed to be a *continuous* random variable[4].

[3]Although the Mittag-Leffler function was introduced by the Swedish mathematician G. Mittag-Leffler in the first years of the twentieth century, it lived for long time in isolation as *Cinderella*. The term *Cinderella function* was used in the fifties by the Italian mathematician F.G. Tricomi for the incomplete gamma function. In recent years the Mittag-Leffler function is gaining more and more popularity in view of the increasing applications of the fractional calculus and is classified as 33E12 in the Mathematics Subject Classification 2000.

[4]The name CTRW became popular in physics after that in the 1960s Montroll, Weiss and Scher (just to cite the pioneers) published a celebrated series of papers on random walks to model diffusion processes on lattices, see *e.g.* [22] and references therein. CTRWs are rather good and general phenomenological models for diffusion, including anomalous diffusion, provided that the resting time of the walker is much greater than the time it takes to make a jump. In fact, in the formalism, jumps are instantaneous. In more recent times, CTRWs were applied back to economics and finance by Hilfer [10], by the authors of the present paper with M. Raberto [20, 14, 9, 18], and, later, by Weiss and co-workers [15].However, it should be noted that the idea of combining a stochastic process for waiting times between two consecutive events and another stochastic process which associates a reward or a claim to each event dates back at least to the first half of the twentieth century with the so-called Cramér–Lundberg model for insurance risk, see for a review [3]. In a probabilistic framework, we now find it more appropriate to refer to all these processes as to *compound renewal processes*.

The position x of the walker at time t is

$$x(t) = x(0) + \sum_{k=1}^{N(t)} X_k \,. \tag{5.1}$$

Let us now denote by $p(x,t)$ the probability density of finding the random walker at the position x at time instant t. We assume the initial condition $p(x,0) = \delta(x)$, meaning that the walker is initially at the origin, $x(0) = 0$. We look for the evolution equation for $p(x,t)$ of the compound renewal process. Based upon the previous probabilistic arguments we arrive at

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \phi(t-t')\, \left[\int_{-\infty}^{+\infty} w(x-x')\, p(x',t')\, dx'\right] dt' \,, \tag{5.2}$$

called the *integral equation of the CTRW*. In fact, from Eq. (5.2) we recognize the role of the survival probability $\Psi(t)$ and of the densities $\phi(t)$, $w(x)$. The first term in the RHS of (5.2) expresses the persistence (whose strength decreases with increasing time) of the initial position $x = 0$. The second term (a space-time convolution) gives the contribution to $p(x,t)$ from the walker sitting in point $x' \in \mathbf{R}$ at instant $t' < t$ jumping to point x just at instant t, after stopping (or waiting) time $t - t'$.

The integral equation (5.2) can be solved by using the machinery of the transforms of Laplace and Fourier. Having introduced the notation for the Laplace transform in sec. 1, we now quote our notation for the Fourier transform, $\mathcal{F}\{f(x);\kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, f(x)\, dx$ ($\kappa \in \mathbf{R}$), and for the corresponding Fourier convolution between (generalized) functions

$$(f_1 \, * \, f_2)\,(x) = \int_{-\infty}^{+\infty} f_1(x')\, f_2(x-x')\, dx' \,.$$

Then, applying the transforms of Fourier and Laplace in succession to (5.2) and using the well-known operational rules, we arrive at the famous Montroll-Weiss equation, see [16],

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{\widetilde{\Psi}(s)}{1 - \widetilde{\phi}(s)\,\widehat{w}(\kappa)} \,. \tag{5.3}$$

As pointed out in [7], this equation can alternatively be derived from the Cox formula, see [2] chapter 8 formula (4), describing the process as subordination of a random walk to a renewal process.

By inverting the transforms one can, in principle, find the evolution $p(x,t)$ of the sojourn density for time t running from zero to infinity. In fact, recalling that $|\widehat{w}(\kappa)| < 1$ and $|\widetilde{\phi}(s)| < 1$, if $\kappa \neq 0$ and $s \neq 0$, Eq. (5.3) becomes

$$\widetilde{\widehat{p}}(\kappa, s) = \widetilde{\Psi}(s) \, \sum_{k=0}^{\infty} [\widetilde{\phi}(s) \, \widehat{w}(\kappa)]^k \, ; \tag{5.4}$$

this gives, inverting the Fourier and the Laplace transforms and taking into account Eqs. (1.9)-(1.10),

$$p(x,t) = \sum_{k=0}^{\infty} P(N(t) = k) \, w_k(x) \, , \tag{5.5}$$

where $w_k(x) = \left(w^{*k}\right)(x)$, in particular $w_0(x) = \delta(x)$, $w_1(x) = w(x)$.

A special case of the integral equation (5.2) is obtained for the *compound Poisson process* where $\phi(t) = \mathrm{e}^{-t}$ (as in (2.1) with $\lambda = 1$ for simplicity). Then, the corresponding equation reduces after some manipulations, that best are carried out in the Laplace-Fourier domain, to the *Kolmogorov-Feller equation*:

$$\frac{\partial}{\partial t} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x - x') \, p(x', t) \, dx' \, , \tag{5.6}$$

which is the *master equation of the compound Poisson process*. In this case, in view of Eqs (2.4) and (5.5) the solution reads

$$p(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \, \mathrm{e}^{-t} \, w_k(x) \, . \tag{5.7}$$

When the survival probability is the Mittag-Leffler function introduced in (3.3), the master equation for the corresponding fractional version of the compound process can be shown to be

$${}_tD_*^{\beta} \, p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x - x') \, p(x', t) \, dx' \, , \quad 0 < \beta < 1 \, , \tag{5.8}$$

where ${}_tD_*^{\beta}$ denotes the time fractional derivative of order β in the Caputo sense. For a (detailed) derivation of Eq (5.8) we refer to the paper by Mainardi et al. [14], in which the results have been obtained by an approach independent from that adopted in a previous paper by Hilfer and Anton [11].

In this case, in view of Eqs (3.10) and (5.5), the solution of the *fractional master equation* (5.8) reads:

$$p(x,t) = \sum_{k=0}^{\infty} \frac{t^{\beta k}}{k!} \, E_\beta^{(k)}(-t^\beta) \, w_k(x) \, . \tag{5.9}$$

In [9] we have, under a power law regime for the jumps, investigated for Eq. (5.8) the so-called *diffusive or hydrodynamic limit*, obtained by making smaller all jumps by a positive factor h and accelerating the process by a large factor properly related to h, then letting h tend to zero. In this limit the master equation (5.8) reduces to a *space-time fractional diffusion equation*. This is also the topic of the recent paper by Scalas et al. [21] and, in a more general framework, of the paper by Gorenflo and Abdel-Rehim [7].

Conclusions

We have provided a *fractional generalization* of the Poisson renewal processes by replacing the first time derivative in the relaxation equation of the survival probability by a fractional derivative of order β $(0 < \beta \leq 1)$. Consequently, we have obtained for $0 < \beta < 1$ non-Markovian renewal processes where, essentially, the exponential probability densities, typical for the Poisson processes, are replaced by functions of Mittag-Leffler type, that decay in a power law manner with an exponent related to β.

The distributions obtained by considering the sum of k *iid* random variables distributed according to the Mittag-Leffler law provide the fractional generalization of the corresponding Erlang distributions. Furthermore, the Mittag-Leffler probability distribution is shown to be the limiting distribution for the thinning procedure of a generic renewal process with waiting time density of power law character.

Then, our theory has been applied to renewal processes *with reward*, so can be considered as the fractional generalization of the compound Poisson processes. In such processes, occurring in time and in space, also the probability distribution of the jump widths is relevant. The stochastic evolution of the space variable in time is modelled by an integro-differential equation (the master equation) which, by containing a time fractional derivative, can be considered as the fractional generalization of the classical Kolmogorov-Feller equation of the compound Poisson process. For this master equation we have provided the analytical solution in terms of iterated derivatives of a Mittag-Leffler function.

Appendix: The Caputo fractional derivative

The *Caputo* fractional derivative provides a fractional generalization of the first derivative through the following rule in the Laplace transform domain,

$$\mathcal{L}\left\{{}_tD_*^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - s^{\beta-1}\, f(0^+)\,, \quad 0 < \beta \le 1\,, \quad s > 0\,, \qquad (A.1)$$

hence turns out to be defined as, see *e.g.* [1, 8],

$${}_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \dfrac{f^{(1)}(\tau)}{(t-\tau)^\beta}\, d\tau\,, & 0 < \beta < 1\,, \\ \dfrac{d}{dt} f(t)\,, & \beta = 1\,. \end{cases} \qquad (A.2)$$

It can alternatively be written in the form

$$\begin{aligned} {}_tD_*^\beta\, f(t) &= \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, d\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)}\, f(0^+) \\ &= \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \int_0^t \frac{f(\tau) - f(0^+)}{(t-\tau)^\beta}\, d\tau\,, \quad 0 < \beta < 1\,. \end{aligned} \qquad (A.3)$$

The Caputo derivative has been indexed with $*$ in order to distinguish it from the classical Riemann-Liouville fractional derivative ${}_tD^\beta$, the first term at the R.H.S. of the first equality in (A.3). As it can be noted from the last equality in (A.3), the Caputo derivative provides a sort of regularization at $t = 0$ of the Riemann-Liouville derivative; for more details see [8].

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