Some Applications of the Fractional Poisson Probability Distribution

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Abstract

New physical and mathematical applications of recently invented fractional Poisson probability distribution have been presented.

As a physical application, a new family of quantum coherent states have been introduced and studied.

Mathematical applications are related to the number theory. We have developed fractional generalization of the Bell polynomials, the Bell numbers, and the Stirling numbers of the second kind.

The fractional Bell polynomials appearance is natural if one evaluates the diagonal matrix element of the evolution operator in the basis of newly introduced quantum coherent states.

The fractional Stirling numbers of the second kind have been introduced and applied to evaluate skewness and kurtosis of the fractional Poisson probability distribution function. A new representation of the Bernoulli numbers in terms of fractional Stirling numbers of the second kind has been found.

In the limit case when the fractional Poisson distribution becomes the well-known Poisson probability distribution all of the above listed new developments and implementations turn into the well-known results of the quantum optics and the number theory.

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1 Introduction

In the past decade it has been realized that the understanding of complex quantum and classical physic phenomena has required the implementation of long-range space and long memory processes.

The mathematical model to capture the long-range space dependence impact on quantum phenomena is the Lévy path integral approach invented and studied in Ref.[1], [2]. The Lévy path integrals generalize the path integral formulation of the quantum mechanics developed in 1948 by Feynman [3], [4].

To study a long memory impact on the counting process, the fractional Poisson random process had been introduced and developed in [5]. The fractional Poisson distribution captures the long-memory effect which results in non-exponential waiting time probability distribution function empirically observed in complex quantum and classical systems. The quantum system example is the fluorescence intermittency of single CdSe quantum dots, that is, the fluorescence emission of single nanocrystals exhibits intermittent behavior, namely, a sequence of "light on" and "light off" states departing from Poisson statistics. In fact, the waiting time distribution in both states is non-exponential [6]. As examples of classical systems let's mention the distribution of waiting times between two consecutive transactions in financial markets [7] and another, which comes from network communication systems, where the duration of network sessions or connections exhibits non-exponential behavior [8].

The non-exponential waiting time distribution function had been obtained for the first time in [9], based on the fractional generalization of the Poisson exponential waiting time distribution.

The fractional Poisson process is a generalization of the well known Poisson process. A simple analytical formula for the fractional Poisson probability distribution function has been obtained in [5] based on the fractional generalization of the Kolmogorov-Feller equation introduced in [9]. Comparing to the standard Poisson distribution, the probability distribution function of the fractional Poisson process [5] has an additional parameter μ, $0 < \mu \leq 1$.

In the limit case $\mu = 1$ the fractional Poisson becomes the standard Poisson process and all our findings are transformed into the well-known results related to the standard Poisson probability distribution.

Now we present quantum physics and the number theory applications of the fractional Poisson probability distribution.

The quantum physics application is an introduction of a new family of quantum coherent states. The motivation to introduce and explore the new coherent states is the observation that the squared modulus $| < n|\varsigma > |^2$ of projection of the newly invented coherent state $|\varsigma >$ onto the eigenstate of the photon number operator $|n >$, gives us the fractional Poisson probability $P_\mu(n)$ that n photons will be found in the new coherent state $|\varsigma >$. Following the Klauder's framework to qualify quantum states as generalized coherent states [10], we prove that our quantum coherent states, (i) are parametrized continuously and normalized; (ii) admit a resolution of unity with positive weight function; (iii) provide temporal stability, that is, the time evolution of coherent states remains within the family of coherent states. We have defined the inner product of two vectors in terms of their coherent state $|\varsigma >$ representation and introduced the functional Hilbert space.

Mathematical applications are related to the number theory. The Bell polynomials, the Bell numbers [11] and the Stirling numbers of the second kind [12] - [14] have been generalized based on the fractional Poisson probability distribution.

Appearance of the fractional Bell polynomials is natural if one evaluates the diagonal matrix element of the quantum evolution operator in the basis of newly introduced quantum coherent states. The appearance of fractional Bell numbers manifests itself in the fractional generalization of the celebrated Dobiñski formula [15], [16] for the generating function of the Bell numbers.

Fractional Stirling numbers of the second kind have been applied to evaluate skewness and kurtosis of the fractional Poisson probability distribution.

In the limit case when $\mu = 1$, the fractional Poisson distribution becomes the well-known Poisson probability distribution, and all above listed new developments and findings turn into the well-known results of the quantum coherent states theory [17], [18], [19] and the number theory [13], [14].

The paper is organized as follows.

The basic definitions of the fractional Poisson random process are briefly reviewed in Sec.2, where Table 1 has been presented to compare formulas related to the fractional Poisson process [5] to those of the well-known ones related to the standard Poisson process. In Sec.3 we introduce and study new quantum coherent states and their applications. Fractional generalizations of the Bell polynomials, Bell numbers and Stirling numbers of the second kind have been introduced and developed in Sec.4. New equations for the generating functions of fractional Bell polynomials, fractional Bell numbers and fractional Stirling numbers of the second kind have been introduced and elaborated. The relationship between the Bernoulli numbers and fractional Stirling numbers of the second kind has been found. Table 3 presents a few fractional Stirling numbers of the second kind. Statistics of the fractional Poisson probability distribution were studied in Sec.5. The centered moment of m-order has been obtained in terms of fractional Stirling numbers of the second kind. Variance, skewness and kurtosis of the fractional Poisson probability distribution function have been presented in terms of the centered moments of m-order. We also explain and discuss how the well-know equations of the quantum optics and the number theory, related to the standard Poisson probability distribution, can be obtained from our generalized results. Tables 2 and 4 summarize the new results related to the fractional Poisson distribution vs those related to the standard Poisson probability distribution.

2 Fundamentals of the fractional Poisson process

The fractional Poisson process has been introduced and developed [5] as the counting process with probability $P_\mu(n,t)$ of arriving n items $(n = 0, 1, 2, ...)$ by time t. The probability $P_\mu(n,t)$ is governed by the fractional generalization of the Kolmogorov-Feller equation

$$ {}_0D_t^\mu P_\mu(n,t) = \nu\left(P_\mu(n-1,t) - P_\mu(n,t)\right) + \frac{t^{-\mu}}{\Gamma(1-\mu)}\delta_{n,0}, \quad 0 < \mu \le 1, \quad (1) $$

with normalization condition

$$ \sum_{n=0}^{\infty} P_\mu(n,t) = 1, \quad (2) $$

where ${}_0D_t^\mu$ is the operator of time derivative of fractional order μ defined as the Riemann-Liouville fractional integral[1],

$$ {}_0D_t^{\mu}\mathrm{f}(t) = \frac{1}{\Gamma(-\mu)} \int_0^t \frac{d\tau \mathrm{f}(\tau)}{(t-\tau)^{1+\mu}}, $$

and $\delta_{n,0}$ is the Kronecker symbol, the gamma function $\Gamma(\mu)$ has the familiar representation $\Gamma(\mu) = \int_0^{\infty} dt e^{-t} t^{\mu-1}$, Re$\mu > 0$, and parameter ν has physical dimension $[\nu] = \sec^{-\mu}$. The initial condition $P_\mu(n, t=0) = \delta_{n,0}$ is incorporated in Eq.(1). One can consider the fractional differential-difference equation (1) as a generalization of the differential-difference equation which defines the standard Poisson process (see, for instance, Eq (1.4.4) in Ref.[23]).

To solve Eq.(1) it is convenient to use the method of the generating function. We introduce the generating function $G_\mu(s,t)$

$$ G_\mu(s,t) = \sum_{n=0}^{\infty} s^n P_\mu(n,t). \tag{3} $$

Then by multiplying Eq.(1) by s^n, summing over n, we obtain the following fractional differential equation for the generating function $G_\mu(s,t)$

$$ {}_0D_t^{\mu} G_\mu(s,t) = \nu \left(\sum_{n=0}^{\infty} s^n P_\mu(n-1,t) - \sum_{n=0}^{\infty} s^n P_\mu(n,t) \right) + \frac{t^{-\mu}}{\Gamma(1-\mu)} = \tag{4} $$

$$ \nu(s-1) G_\mu(s,t) + \frac{t^{-\mu}}{\Gamma(1-\mu)}. $$

The solution of this fractional equation has a form

$$ G_\mu(s,t) = E_\mu(\nu t^{\mu}(s-1)), \tag{5} $$

where $E_\mu(z)$ is the Mittag-Leffler function given by its power series [24], [25][2]

$$ E_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\mu m + 1)}. \tag{6} $$

[1]The basic formulas on fractional calculus can be found in Refs. [20] - [22].

[2]At $\mu = 1$ the function $E_\mu(z)$ turns into $\exp(z)$.

Expanding (5) in series over s results in accordance with definition (3)

$$P_\mu(n,t) = \frac{(\nu t^\mu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))}, \qquad 0 < \mu \le 1. \tag{7}$$

The $P_\mu(n,t)$ gives us the probability that in the time interval $[0,t]$ we observe n counting events. When $\mu = 1$ the $P_\mu(n,t)$ is transformed to the standard Poisson distribution (see Eq.(14) in Ref.[5]). Thus, Eq.(7) can be considered as a fractional generalization of the standard Poisson probability distribution. The presence of an additional parameter μ brings new features in comparison with the standard Poisson distribution.

On a final note, the probability distribution of the fractional Poisson process can be represented in terms of the Mittag-Leffler function $E_\mu(z)$ in the following compact way [5],

$$P_\mu(n,t) = \frac{(-z)^n}{n!} \frac{d^n}{dz^n} E_\mu(z)|_{z=-\nu t^\mu} \tag{8}$$

$$P_\mu(n=0,t) = E_\mu(-\nu t^\mu). \tag{9}$$

At $\mu = 1$ Eqs.(8) and (9) are transformed into the well known equations for the standard Poisson process with the substitution $\nu \to \overline{\nu}$, where $\overline{\nu}$ is the rate of arrivals of the standard Poisson process with physical dimension $\overline{\nu} = \sec^{-1}$.

Table 1 summarizes equations attributed to the fractional Poisson process with those belonging to the well-known standard Poisson process. Table 1 presents two sets of equations for the probability distribution function $P(n,t)$ of n events having arrived by time t, the probability $P(0,t)$ of nothing having arrived by time t, mean $\overline{n}$, variance σ^2, generating function $G(s,t)$ for the probability distribution function, moment generating function $H(s,t)$, and waiting-time probability distribution function $\psi(\tau)$.

	fractional Poisson ($0 < \mu \leq 1$)	Poisson ($\mu = 1$)
$P(n,t)$	$\frac{(\nu t^\mu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))}$	$\frac{(\overline{\nu} t)^n}{n!} \exp(-\overline{\nu} t)$
$P(n,t)$	$\frac{(-z)^n}{n!} \frac{d^n}{dz^n} E_\mu(z)\|_{z=-\nu t^\mu}$	$\frac{(-z)^n}{n!} \frac{d^n}{dz^n} \exp(z)\|_{z=-\overline{\nu} t^\mu}$
$P(0,t)$	$E_\mu(-\nu t^\mu)$	$\exp(-\overline{\nu} t)$
$\overline{n}$	$\frac{\nu t^\mu}{\Gamma(\mu+1)}$	$\overline{\nu} t$
σ^2	$\frac{\nu t^\mu}{\Gamma(\mu+1)} + \left(\frac{\nu t^\mu}{\Gamma(\mu+1)}\right)^2 \left\{\frac{\mu B(\mu,\frac{1}{2})}{2^{2\mu-1}} - 1\right\}$	$\overline{\nu} t$
$G(s,t)$	$E_\mu(\nu t^\mu(s-1))$	$\exp\{\overline{\nu} t(s-1)\}$
$H(s,t)$	$E_\mu(\nu t^\mu(e^{-s}-1))$	$\exp\{\overline{\nu} t(e^{-s}-1)\}$
$\psi(\tau)$	$\nu \tau^{\mu-1} E_{\mu,\mu}(-\nu \tau^\mu)$	$\overline{\nu} e^{-\overline{\nu}\tau}$

Table 1. *Fractional Poisson process vs the Poisson process*[3].

3 New family of coherent states

The quantum mechanical states first introduced by Schrödinger [26] to study the quantum harmonic oscillator are now well-known as the coherent states. Coherent states provide an important theoretical paradigm to study electromagnetic field coherence and quantum optics phenomena [17], [18].

The standard coherent states are given for all complex numbers $z \in \mathbb{C}$, by

$$|z> = e^{(za^+ - z^* a)}|0> = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n>, \tag{10}$$

where a^+ and a are photon field creation and annihilation operators that satisfy the Bose-Einstein commutation relation $[a, a^+] = aa^+ - a^+a = 1$, and the orthonormal vector $|n> = \frac{1}{\sqrt{n!}}(a^+)^n|0>$, which is an eigenvector of the photon number operator $N = a^+a$, $N|n> = n|n>$, $<n|n'> = \delta_{n,n'}$.

The projection of the coherent state $|z>$ onto the state $|n>$ is

$$<n|z> = \frac{z^n}{\sqrt{n!}} e^{-\frac{1}{2}|z|^2}. \tag{11}$$

[3]All definitions and equations related to the fractional Poisson process are taken from [5].

Then the squared modulus of $< n|z >$ gives us the probability $P(n)$ that n photons will be found in the coherent state $|z>$. Thus, we come to the well-know result for probability $P(n)$

$$P(n) = | < n|z > |^2 = \frac{|z|^{2n}}{n!} e^{-|z|^2}, \tag{12}$$

which is recognized as a Poisson probability distribution with a mean value $|z|^2$. The value $|z|^2$ is in fact the mean number of photons when the state is a coherent state $|z>$

$$|z|^2 = \sum_{n=0}^{\infty} nP(n) = < z|a^+a|z > . \tag{13}$$

We introduce a new family of coherent states $|\varsigma>$

$$|\varsigma >= \sum_{n=0}^{\infty} \frac{(\sqrt{\mu}\varsigma^{\mu})^n}{\sqrt{n!}} (E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2}|n >, \tag{14}$$

and adjoint states $< \varsigma|$

$$< \varsigma| = \sum_{n=0}^{\infty} < n| \frac{(\sqrt{\mu}\varsigma^{*\mu})^n}{\sqrt{n!}} (E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2}, \tag{15}$$

where

$$E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu}) = \frac{d^n}{dx^n} E_{\mu}(x)|_{x=-\mu|\varsigma|^{2\mu}} \tag{16}$$

and $E_{\mu}(x)$ is the Mittag-Leffler function defined by Eq.(6), complex number ς stands for labelling the new coherent states, and the orthonormal vector $|n>$ is the same as for Eq.(10).

To motivate the introduction of new coherent states $|\varsigma>$ we calculate the projection of the coherent state $|\varsigma>$ onto the state $|n>$,

$$< n|\varsigma >= \frac{(\sqrt{\mu}\varsigma^{\mu})^n}{\sqrt{n!}} \left(E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu})\right)^{1/2},$$

then the squared modulus of $< n|\varsigma >$ gives us the probability $P_{\mu}(n)$ that n photons will be found in the quantum coherent state $|\varsigma>$. Thus, we come to the fractional Poisson probability distribution of photon numbers $P_{\mu}(n)$

$$P_{\mu}(n) = | < n|\varsigma > |^2 = \frac{(\mu|\varsigma|^{2\mu})^n}{n!} \left(E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu})\right), \tag{17}$$

with mean value $(\mu|\varsigma|^{2\mu})/\Gamma(\mu+1)$. The value $(\mu|\varsigma|^{2\mu})/\Gamma(\mu+1)$ is in fact the mean number of photons when the quantum state is the coherent state $|\varsigma>$

$$(\mu|\varsigma|^{2\mu})/\Gamma(\mu+1) = \sum_{n=0}^{\infty} nP_{\mu}(n) =< \varsigma|a^{+}a|\varsigma> .$$

It is easy to see that when $\mu = 1$ we have $E_1(x) = \exp(-x)$ and $E_{\mu}^{(n)}(-x) = \exp(-x)$. Hence, Eq. (14) turns into to Eq.(10) and Eq. (17) leads to Eq.(11). In other words, the new coherent states defined by Eq.(14) generalize the standard coherent states Eq.(10), and this generalization has been implemented with the help of the fractional Poisson probability distribution. The new family of coherent states Eq.(14) has been designed here to study physical phenomena where the distribution of photon numbers is governed by the fractional Poisson distribution Eq.(17).

Let's answer the question if the newly introduced coherent states $|\varsigma>$ are really generalized coherent states?

Quantum mechanical states are generalized coherent states if they [10]:

(i) are parametrized continuously and normalized;

(ii) admit a resolution of unity with a positive weight function;

(iii) provide temporal stability, that is, the time evolving coherent state belongs to the family of coherent states.

Let's now show that the new coherent states introduced by Eq.(14) satisfy all above listed conditions.

To prove (i), we note that the coherent states $|\varsigma>$ introduced by Eq.(14) are evidently parametrized continuously by their label ς which is a complex number $\varsigma = \xi + i\eta$, with $\xi = \mathrm{Re}\varsigma$ and $\eta = \mathrm{Im}\varsigma$. Because of the normalization condition of the fractional Poisson probability distribution $\sum_{n=0}^{\infty} P_{\mu}(n) = 1$ and $< n|n'>= \delta_{n,n'}$, we have

$$< \varsigma|\varsigma>=$$

$$\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} < n|\frac{(\sqrt{\mu}\varsigma^{*\mu})^{n}}{\sqrt{n!}}(E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2}\frac{(\sqrt{\mu}\varsigma^{\mu})^{n'}}{\sqrt{n'!}}(E_{\mu}^{(n')}(-\mu|\varsigma|^{2\mu}))^{1/2}|n'>= \tag{18}$$

$$\sum_{n=0}^{\infty} \frac{(\mu|\varsigma|^{2\mu})^n}{n!} \left(E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu})\right) = \sum_{n=0}^{\infty} P_{\mu}(n) = 1,$$

that is, the coherent states $|\varsigma>$ are normalized.

To prove (ii), that is, the coherent states $|\varsigma>$ admit a resolution of unity with a positive weight function, we introduce a positive function $W_{\mu}(|\varsigma|^2) > 0$ which obeys the equation

$$\frac{1}{\pi} \int\limits_{\mathbb{C}} d^2\varsigma |\varsigma > W_{\mu}(|\varsigma|^2) < \varsigma| = I, \tag{19}$$

where $d^2\varsigma = d(\mathrm{Re}\varsigma)d(\mathrm{Im}\varsigma)$ and the integration extends over the entire complex plane $\mathbb{C}$. This equation with yet unknown function $W_{\mu}(|\varsigma|^2)$ can be considered as "a resolution of unity". To find the function $W_{\mu}(|\varsigma|^2)$ let's transform Eq.(19). Introducing of new integration variables ρ and ϕ by $\varsigma = \rho e^{i\phi}$, $d^2\varsigma = \rho^2 d\rho d\phi$ and making use of Eqs.(14) and (15) yield

$$\frac{1}{\pi} \int\limits_{\mathbb{C}} d^2\varsigma |\varsigma > W_{\mu}(|\varsigma|^2) < \varsigma| =$$

$$\frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int\limits_0^{\infty} d\rho \rho^{(n+m)\mu+1} \int\limits_0^{2\pi} d\phi e^{i(n-m)\phi} \frac{W_{\mu}(\rho^2)}{\sqrt{n!m!}} \left(E_{\mu}^{(n)}(-\mu|\rho|^{2\mu})\right)^{1/2} \times \tag{20}$$

$$\left(E_{\mu}^{(m)}(-\mu|\rho|^{2\mu})\right)^{1/2} |n >< m| = I.$$

By interchanging the orders of summation and integration and carrying out the integration over ϕ, we get a factor $2\pi\delta_{n,m}$, which reduces the double summation to a single one. Therefore, Eq.(20) is simplified to

$$\frac{1}{\pi} \int\limits_{\mathbb{C}} d^2\varsigma |\varsigma > W_{\mu}(|\varsigma|^2) < \varsigma| =$$

$$\sum_{n=0}^{\infty} \frac{2}{n!} \int\limits_0^{\infty} d\rho \rho^{2n\mu+1} W_{\mu}(\rho^2) E_{\mu}^{(n)}(-\mu|\rho|^{2\mu}) |n >< n| = I. \tag{21}$$

Because of the completeness of orthonormal vectors $|n>$

$$\sum_{n=0}^{\infty} |n><n| = I, \tag{22}$$

we come to the following integral equation to find the positive function $W_\mu(x)$

$$\int_0^\infty dx x^{n\mu} W_\mu(x) E_\mu^{(n)}(-\mu x^\mu) = n!. \tag{23}$$

To solve Eq.(23) we use the Laplace transform of the function $t^{\mu n} E_\mu^{(n)}(-\mu t^\mu)$, see Appendix,

$$\int_0^\infty dt e^{-st} t^{\mu n} E_\mu^{(n)}(-\mu t^\mu) = \frac{n! \cdot s^{\mu-1}}{(s^\mu + \mu)^{n+1}}. \tag{24}$$

By comparing Eqs.(23) and (24) we conclude that the positive function $W_\mu(x)$ has the form

$$W_\mu(x) = (1-\mu)^{\frac{1-\mu}{\mu}} \cdot \exp\{-(1-\mu)^{1/\mu} x\}, \qquad 0 < \mu \leq 1. \tag{25}$$

Thus, we proved that the coherent states $|\varsigma>$ admit a resolution of unity with the positive weight function $W_\mu(x)$ given by Eq.(25). At $\mu = 1$, function $W_\mu(x)$ becomes

$$W_1(x) = \lim_{\mu \to 1} W_\mu(x) = \lim_{\mu \to 1} \left[(1-\mu)^{\frac{1-\mu}{\mu}} \cdot \exp\{-(1-\mu)^{1/\mu} x\} \right] = 1,$$

and we come back to the resolution of unity equation for the standard coherent states $|z>$

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2 z |z><z| = I. \tag{26}$$

To prove (iii), we note that if $|n>$ is an eigenvector of the Hamiltonian $H = \hbar\omega N = \hbar\omega a^+ a$, where $\hbar$ is Planck's constant, then the time evolution operator $\exp(-iHt/\hbar)$ results

$$\exp(-iHt/\hbar)|n>= e^{-i\omega n t}|n> .$$

In other words, the time evolution of $|n>$ results in appearance of the phase factor only while the state does not change. Let's consider time evolution of the coherent state $|\varsigma>$ defined by Eq.(14). Well, as far as the coherent state is not an eigenstate of H then one may expect that it evolves into other states in time. However, it follows that

$$\exp(-iHt/\hbar)|\varsigma>=\sum_{n=0}^{\infty}\frac{(\sqrt{\mu}\varsigma^{\mu})^n}{\sqrt{n!}}(E_{\mu}^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2}e^{-i\omega nt}|n>=|e^{-\frac{i\omega t}{\mu}}\varsigma>, \tag{27}$$

which is just another coherent state belonging to a complex eigenvalue $\varsigma e^{-\frac{i\omega t}{\mu}}$. We see that the time evolution of the coherent state $|\varsigma>$ remains within the family of coherent states $|\varsigma>$. The property embodied in Eq.(27) is the temporal stability of coherent states $|\varsigma>$ under the action of H.

Thus, we conclude that the new coherent states $|\varsigma>$ satisfy the Klauder's criteria set (i) - (iii) for generalized coherent states.

Finally, let us introduce an alternative notation for $|\varsigma>$ in terms of the real q and imaginary p parts of $|\varsigma>$, that is, $|\varsigma>=(q+ip)/\sqrt{2\hbar}$. Then from Eq.(14) we have

$$|\varsigma>=|p,q>=\sum_{n=0}^{\infty}\frac{(\sqrt{\mu}(q+ip)^{\mu})^n}{(2\hbar)^{n/2}\sqrt{n!}}(E_{\mu}^{(n)}(-\mu\left(\frac{q^2+p^2}{4\hbar}\right)^{\mu}))^{1/2}|n>. \tag{28}$$

The adjoint coherent states are defined as

$$<\varsigma|=<p,q|=\sum_{n=0}^{\infty}<n|\frac{(\sqrt{\mu}(q-ip)^{\mu})^n}{(2\hbar)^{n/2}\sqrt{n!}}(E_{\mu}^{(n)}(-\mu\left(\frac{q^2+p^2}{4\hbar}\right)^{\mu}))^{1/2}. \tag{29}$$

Despite the fact that the adjoint state is labelled by ς, the series expansion Eq.(29) are formed in fact of powers of ς^*.

3.1 Quantum mechanical vector and operator representations based on coherent states $|\varsigma>$

In spirit of Klauder's consideration [17], let's show that the resolution of unity criteria Eq.(19) with $W_{\mu}(x)$ given by Eq.(25) allows us to list fundamental quantum mechanical statements pertaining to the associated representation of Hilbert space. Indeed, it is easy to see that the newly introduced coherent states $|\varsigma>$ provide:

1. Inner Product of quantum mechanical vectors $|\varphi>$ and $|\psi>$ defined as

$$<\varphi|\psi>=\frac{1}{\pi}\int\limits_{\mathbb{C}} d^2\varsigma <\varphi|\varsigma> W_\mu(|\varsigma|^2) <\varsigma|\psi>, \tag{30}$$

where the vector representatives are wave functions $<\varphi|\varsigma>$ and $<\varsigma|\psi>$ given by

$$<\varphi|\varsigma>=\sum_{n=0}^{\infty}\frac{(\sqrt{\mu}\varsigma^\mu)^n}{\sqrt{n!}}(E_\mu^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2} <\varphi|n>, \tag{31}$$

$$<\varsigma|\psi>=\sum_{n=0}^{\infty} <n|\psi> \frac{(\sqrt{\mu}\varsigma^{*\mu})^n}{\sqrt{n!}}(E_\mu^{(n)}(-\mu|\varsigma|^{2\mu}))^{1/2}. \tag{32}$$

2. Vectors Transformation Law

$$<\varsigma|\mathcal{A}|\psi>=\frac{1}{\pi}\int\limits_{\mathbb{C}} d^2\varsigma^{'} <\varsigma|\mathcal{A}|\varsigma^{'}> W_\mu(|\varsigma^{'}|^2) <\varsigma^{'}|\psi>, \tag{33}$$

where $<\varsigma|\mathcal{A}|\varsigma^{'}>$ is the matrix element of quantum mechanical operator $\mathcal{A}$.

3. Operator Transformation Law

$$<\varsigma|\mathcal{A}_1\mathcal{A}_2|\varsigma^{'}>=\frac{1}{\pi}\int\limits_{\mathbb{C}} d^2\varsigma^{''} <\varsigma|\mathcal{A}_1|\varsigma^{''}> W_\mu(|\varsigma^{''}|^2) <\varsigma^{''}|\mathcal{A}_2|\varsigma^{''}>, \tag{34}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two quantum mechanical operators.

Further, the inverse map from the functional Hilbert space representation of coherent states $|\varsigma>$ to the abstract one is provided by the following decomposition laws:

4. Vector Decomposition Law

$$|\psi>=\frac{1}{\pi}\int\limits_{\mathbb{C}} d^2\varsigma\cdot|\varsigma> W_\mu(|\varsigma|^2) <\varsigma|\psi>. \tag{35}$$

5. Operator Decomposition Law

$$\mathcal{A} = \frac{1}{\pi} \int\limits_{\mathbb{C}} d^2\varsigma_1 d^2\varsigma_2 \cdot |\varsigma_1 > W_\mu(|\varsigma_1|^2) < \varsigma_1|\mathcal{A}|\varsigma_2 > W_\mu(|\varsigma_2|^2) < \varsigma_2|. \qquad (36)$$

Thus, we conclude that the resolution of unity Eq.(19) with $W_\mu(x)$, given by Eq.(25), provides an appropriate inner product Eq.(30) and lets us introduce the Hilbert space.

All of the above listed results lead to the well-know fundamental equations of quantum optics and coherent states theory [17], [18] in the limit case $\mu = 1$.

Table 2 summarizes the definitions and equations attributed to the newly introduced coherent states $|\varsigma >$ with those for the coherent states $|z >$. Table 2 presents two sets of equations for a coherent state $|... >$, for an adjoint coherent state $<...|$, for the probability $P(n)$ that n photons will be found in the coherent state $|... >$, for a positive weight function $W(x)$ in the resolution of unity equations (19) and (26), the mean number $< ...|a^+a|... >$ of photons when the state is a coherent state $|... >$, and the quantum mechanical vector $|\psi >$ decomposition law.

	$\|\varsigma > (0 < \mu \leq 1)$	$\|z >$
$\|... >$	$\sum\limits_{n=0}^{\infty} \frac{(\sqrt{\mu}\varsigma^\mu)^n}{\sqrt{n!}} (E_\mu^{(n)}(-\mu\|\varsigma\|^{2\mu}))^{1/2}\|n >$	$e^{-\frac{1}{2}\|z\|^2} \sum\limits_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\|n >$
$<...\|$	$\sum\limits_{n=0}^{\infty} < n\| \frac{(\sqrt{\mu}\varsigma^{*\mu})^n}{\sqrt{n!}} (E_\mu^{(n)}(-\mu\|\varsigma\|^{2\mu}))^{1/2}$	$<n\|e^{-\frac{1}{2}\|z\|^2} \sum\limits_{n=0}^{\infty} \frac{z^{*n}}{\sqrt{n!}}$
$P(n)$	$\frac{(\mu\|\varsigma\|^{2\mu})^n}{n!} E_\mu^{(n)}(-\mu\|\varsigma\|^{2\mu})$	$\frac{\|z\|^{2n}}{n!} e^{-\|z\|^2}$
$W(x)$	$(1-\mu)^{\frac{1-\mu}{\mu}} \cdot \exp\{-(1-\mu)^{\frac{1}{\mu}} x\}$	1
$< ...\|a^+a\|... >$	$(\mu\|\varsigma\|^{2\mu})/\Gamma(\mu+1)$	$\|z\|^2$
$\|\psi >$	$\frac{1}{\pi} \int\limits_{\mathbb{C}} d^2\varsigma\|\varsigma > W_\mu(\|\varsigma\|^2) < \varsigma\|\psi >$	$\frac{1}{\pi} \int\limits_{\mathbb{C}} d^2z\|z >< z\|\psi >$

Table 2. *Coherent states $|\varsigma >$ vs coherent states $|z >$.*

4 Generalized Bell and Stirling Numbers

4.1 Fractional Bell polynomials and Bell numbers

Based on the fractional Poisson probability distribution Eq.(7) we introduce a new generalization of the Bell polynomials

$$B_\mu(x,m) = \sum_{n=0}^{\infty} n^m \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-x)^k}{\Gamma(\mu(k+n)+1)}, \qquad B_\mu(x,0)=1, \tag{37}$$

where the parameter μ is $0 < \mu \leq 1$. We will call $B_\mu(x,m)$ as the fractional Bell polynomials.

The polynomials $B_\mu(x,m)$ are related to the well-known Bell polynomials [11] $B(x,m)$ by

$$B_\mu(x,m)|_{\mu=1} = B(x,m) = e^{-x} \sum_{n=0}^{\infty} n^m \frac{x^n}{n!}. \tag{38}$$

From Eq.(37) we come to a new formula for the numbers $B_\mu(m)$, which we call the fractional Bell numbers

$$B_\mu(m) = B_\mu(x,m)|_{x=1} = \sum_{n=0}^{\infty} \frac{n^m}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-1)^k}{\Gamma(\mu(k+n)+1)} \tag{39}$$

or

$$B_\mu(m) = \sum_{n=0}^{\infty} (-1)^n \frac{n^m}{n!} E_\mu^{(n)}(-1), \tag{40}$$

where $E_\mu^{(n)}(-x)$ is defined by Eq.(6). The new formula Eq.(40) is in fact a fractional generalization of the so-called Dobiñski relation known since 1877 [15], [16]. Indeed, at $\mu = 1$ when the Mittag-Leffler function is just the exponential function, $E_1(x) = \exp(-x)$, we have, $(-1)^n E_1^{(n)}(-1) = e^{-1}$, and the new equation (40) becomes the Dobiñski relation [15] for the Bell numbers $B(m)$,

$$B(m) = B_\mu(m)|_{\mu=1} = e^{-1} \sum_{n=0}^{\infty} \frac{n^m}{n!}. \tag{41}$$

As an example, here are a few fractional Bell numbers

$$B_\mu(0) = 1, \qquad B_\mu(1) = \frac{1}{\Gamma(\mu+1)}, \qquad B_\mu(2) = \frac{2}{\Gamma(2\mu+1)} + \frac{1}{\Gamma(\mu+1)},$$

$$B_\mu(3) = \frac{6}{\Gamma(3\mu+1)} + \frac{6}{\Gamma(2\mu+1)} + \frac{1}{\Gamma(\mu+1)},$$

$$B_\mu(4) = \frac{24}{\Gamma(4\mu+1)} + \frac{36}{\Gamma(3\mu+1)} + \frac{14}{\Gamma(2\mu+1)} + \frac{1}{\Gamma(\mu+1)}.$$

Now we focus on the new general definitions given by Eqs.(37) and (40) to find the generating functions of the polynomials $B_\mu(x,m)$ and the numbers $B_\mu(m)$. Let us introduce the generating function of the new polynomials $B_\mu(x,m)$ as

$$F_\mu(s,x) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_\mu(x,m). \tag{42}$$

Therefore, to get the polynomial $B_\mu(x,m)$ we should differentiate $F_\mu(s,x)$ m times with respect to s, and then let $s=0$. That is,

$$B_\mu(x,m) = \frac{\partial^m}{\partial s^m} F_\mu(s,x)|_{s=0}. \tag{43}$$

To find an explicit equation for $F_\mu(s,x)$, let's substitute Eq.(37) into Eq.(42) and evaluate the sum over m. As a result we have

$$F_\mu(s,x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} e^{sn} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-x)^k}{\Gamma(\mu(k+n)+1))}. \tag{44}$$

Then, introducing the new summation variable $l = k+n$, yields

$$F_\mu(s,x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} e^{sn} \sum_{l=n}^{\infty} \frac{l!}{(l-n)!} \frac{(-x)^{l-n}}{\Gamma(\mu l+1)} =$$

$$\sum_{l=0}^{\infty} \frac{1}{\Gamma(\mu l+1)} \sum_{n=0}^{l} \frac{l!}{n!(l-n)!} e^{sn} x^n (-x)^{l-n} = \sum_{l=0}^{\infty} \frac{(xe^s - x)^l}{\Gamma(\mu l+1)}.$$

Finally, we obtain

$$F_\mu(s,x) = E_\mu(x(e^s - 1)), \tag{45}$$

where $E_\mu(z)$ is the Mittag-Leffler function given by the power series Eq.(6).

It is easy to see that the generating function $F_\mu(s,x)$, given by Eq.(45), can be considered as the moment generating function of the fractional Poisson probability distribution (see Eq.(35) in Ref.[5]).

In the case of $\mu = 1$, Eq.(45) turns into the equation for the generating function of the Bell polynomials,

$$F_1(s,x) = \exp\{(x(e^s - 1))\}. \tag{46}$$

If we put $x = 1$ in Eq.(45), then we immediately come to the generating function $\mathcal{B}_\mu(s)$ of the fractional Bell numbers $B_\mu(m)$

$$\mathcal{B}_\mu(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_\mu(m) = E_\mu(e^s - 1). \tag{47}$$

The numbers $B_\mu(m)$ can be obtained by differentiating $\mathcal{B}_\mu(s)$ m times with respect to s, and then letting $s = 0$,

$$B_\mu(m) = \frac{\partial^m}{\partial s^m} \mathcal{B}_\mu(s,x)|_{s=0}. \tag{48}$$

When $\mu = 1$, Eq.(47) reads

$$\mathcal{B}_1(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_1(m) = \exp\{(e^s - 1)\}, \tag{49}$$

and we come to the well-known equation for the Bell numbers generating function.

Let us show how the fractional Bell polynomials are related to the new coherent states $|\varsigma>$ introduced by Eq.(14). For the boson creation a^+ and a annihilation operators of a photon field that satisfy the commutation relation $[a, a^+] = aa^+ - a^+a = 1$, the diagonal matrix element of the n-th power of the number operator $(a^+a)^n$ yields the fractional Bell polynomials of order n,

$$< \varsigma|(a^+a)^n|\varsigma >= B_\mu(|\varsigma|^2, n). \tag{50}$$

Then, the diagonal coherent state $|\varsigma>$ matrix element $<\varsigma|\exp(-iHt/\hbar)|\varsigma>$ of the time evolution operator

$$\exp(-iHt/\hbar) = \exp\left\{(-i\omega t/\hbar)a^{+}a\right\} \tag{51}$$

can be written as

$$<\varsigma|\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}|\varsigma>= \sum_{n=0}^{\infty}\frac{1}{n!}(-i\frac{\omega t}{\hbar})^{n}<\varsigma|(a^{+}a)^{n}|\varsigma>= \tag{52}$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}(-i\frac{\omega t}{\hbar})^{n}B_{\mu}(|\varsigma|^{2},n),$$

where $\hbar$ is Planck's constant.

By comparing with Eq.(42) we conclude that

$$<\varsigma|\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}|\varsigma>= E_{\mu}(|\varsigma|^{2}(\exp(-i\omega t/\hbar)-1)). \tag{53}$$

In other words, the diagonal coherent state $|\varsigma>$ matrix element of the operator $\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}$ is the generating function of the fractional Bell polynomials. In the special case $\mu = 1$, Eq.(53) reads

$$<z|\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}|z>= \exp\{(|z|^{2}(\exp(-i\omega t/\hbar)-1)\}, \tag{54}$$

that is, the diagonal coherent state $|z>$ matrix element of the operator $\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}$ is the generating function of the Bell polynomials. This statement immediately follows from Eqs.(11.5-2) and (11.2-10) of Ref.[19] for the diagonal matrix element of the operator $\exp\left\{(-i\omega t/\hbar)a^{+}a\right\}$ in the basis of the coherent states $|z>$.

4.2 Fractional Stirling numbers of the second kind

Now we are set up to introduce the fractional generalization of the Stirling numbers[4] of the second kind $S_{\mu}(m,l)$ by means of equation

[4]Stirling numbers, introduced by J. Stirling [12] in 1730, have been studied in the past by many celebrated mathematicians. Among them are Euler, Lagrange, Laplace and Cauchy. Stirling numbers play an important role in combinatorics, number theory, probability and statistics. There are two common sets of Stirling numbers, they are so-called Stirling numbers of the first kind and Stirling numbers of the second kind (for details, see Chapter 8 in Ref. [13] and [14]).

$$B_\mu(x,m) = \sum_{l=0}^{m} S_\mu(m,l) x^l, \tag{55}$$

where $B_\mu(x,m)$ is a fractional generalization of the Bell polynomials given by Eq.(37) and the parameter μ is $0 < \mu \leq 1$. At $\mu = 1$, Eq.(55) defines the integers $S(m,l) = S_\mu(m,l)|_{\mu=1}$, which are called Stirling numbers of the second kind. At $x = 1$, when the fractional Bell polynomials $B_\mu(x,m)$ become the fractional Bell numbers, $B_\mu(m) = B_\mu(x,m)|_{x=1}$, Eq.(55) gives us a new equation to express fractional Bell numbers in terms of fractional Stirling numbers of the second kind

$$B_\mu(m) = \sum_{l=0}^{m} S_\mu(m,l). \tag{56}$$

To find $S_\mu(m,l)$ we transform the right hand side of Eq.(37) as follows

$$B_\mu(x,m) = \sum_{n=0}^{\infty} n^m \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-x)^k}{\Gamma(\mu(k+n)+1))} =$$

$$\sum_{n=0}^{\infty} n^m \frac{x^n}{n!} \sum_{l=0}^{\infty} \theta(l-n) \frac{l!}{(l-n)!} \frac{(-x)^{l-n}}{\Gamma(\mu l+1)}, \tag{57}$$

here $\theta(l)$ is the Heaviside step function,

$$\theta(l) = \left| \begin{array}{ll} 1, & \text{if } l \geq 0 \\ 0, & \text{if } l < 0 \end{array} \right. . \tag{58}$$

Then, interchanging the order of summations in Eq.(57) yields

$$B_\mu(x,m) = \sum_{l=0}^{\infty} \frac{x^l}{\Gamma(\mu l+1)} \sum_{n=0}^{l} n^m \frac{(-1)^{l-n} l!}{n!(l-n)!} = \sum_{l=0}^{\infty} \frac{x^l}{\Gamma(\mu l+1)} \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m, \tag{59}$$

where the notation $\binom{l}{n} = \frac{l!}{n!(l-n)!}$ has been introduced.

By comparing Eq.(55) and Eq.(59) we conclude that the fractional Stirling numbers $S_\mu(m,l)$ are given by

$$S_\mu(m,l) = \frac{1}{\Gamma(\mu l+1)} \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m, \tag{60}$$

$$S_\mu(m,0) = \delta_{m,0}, \qquad S_\mu(m,l) = 0, \quad l = m+1, \quad m+2, ...$$

As an example, Table 3 presents a few of fractional Stirling numbers of the second kind.

m\\l	1	2	3	4	5	6	7
1	$\frac{1}{\Gamma(\mu+1)}$						
2	$\frac{1}{\Gamma(\mu+1)}$	$\frac{2}{\Gamma(2\mu+1)}$					
3	$\frac{1}{\Gamma(\mu+1)}$	$\frac{6}{\Gamma(2\mu+1)}$	$\frac{6}{\Gamma(3\mu+1)}$				
4	$\frac{1}{\Gamma(\mu+1)}$	$\frac{14}{\Gamma(2\mu+1)}$	$\frac{36}{\Gamma(3\mu+1)}$	$\frac{24}{\Gamma(4\mu+1)}$			
5	$\frac{1}{\Gamma(\mu+1)}$	$\frac{30}{\Gamma(2\mu+1)}$	$\frac{150}{\Gamma(3\mu+1)}$	$\frac{240}{\Gamma(4\mu+1)}$	$\frac{120}{\Gamma(5\mu+1)}$		
6	$\frac{1}{\Gamma(\mu+1)}$	$\frac{62}{\Gamma(2\mu+1)}$	$\frac{540}{\Gamma(3\mu+1)}$	$\frac{1560}{\Gamma(4\mu+1)}$	$\frac{1800}{\Gamma(5\mu+1)}$	$\frac{720}{\Gamma(6\mu+1)}$	
7	$\frac{1}{\Gamma(\mu+1)}$	$\frac{126}{\Gamma(2\mu+1)}$	$\frac{1806}{\Gamma(3\mu+1)}$	$\frac{8400}{\Gamma(4\mu+1)}$	$\frac{16800}{\Gamma(5\mu+1)}$	$\frac{15120}{\Gamma(6\mu+1)}$	$\frac{5040}{\Gamma(7\mu+1)}$

Table 3. *Fractional Stirling numbers of the second kind* $S_\mu(m,l)$ $(0 < \mu \le 1)$.

Some special cases are

$$S_\mu(n,1) = \frac{1}{\Gamma(\mu+1)}, \qquad S_\mu(n,2) = (2^{n-1}-1)\frac{2}{\Gamma(2\mu+1)}, \tag{61}$$

$$S_\mu(n,3) = (3^n - 3\cdot 2^n + 3)\frac{1}{\Gamma(3\mu+1)}, \tag{62}$$

$$S_\mu(n,4) = (4^n - 4\cdot 3^n + 6\cdot 2^n - 4)\frac{1}{\Gamma(4\mu+1)}, \tag{63}$$

$$S_\mu(n,n-1) = \frac{n!\cdot(n-1)}{2\Gamma((n-1)\mu+1)}, \qquad S_\mu(n,n) = \frac{n!}{\Gamma(n\mu+1)}. \tag{64}$$

It is easy to see that at $\mu = 1$ the above equation turns into the well known representation for the standard Stirling numbers $S(m,l) \equiv S_1(m,l)$ of the second kind [27],

$$S(m,l) = \frac{1}{l!}\sum_{n=0}^{l}(-1)^{l-n}\binom{l}{n}n^m.$$

Thus, one can conclude that there is a relationship between fractional Stirling numbers $S_\mu(m,l)$ of the second kind and standard Stirling numbers $S(m,l)$ of the second kind

$$S_\mu(m,l) = \frac{l!}{\Gamma(\mu l+1)} S(m,l). \tag{65}$$

or

$$S(m,l) = \frac{\Gamma(\mu l+1)}{l!} S_\mu(m,l). \tag{66}$$

Let's note that Eqs.(65) or (66) allow us to find new equations and identities for the fractional Stirling numbers $S_\mu(m,l)$ based on the well-know equations and identities for the standard Stirling numbers $S(m,l)$.

To find a generating function of the fractional Stirling numbers $S_\mu(m,l)$, let's expand the generating function $F_\mu(s,x)$ given by Eq.(42). Upon substituting $B_\mu(x,m)$ from Eq.(55) we have the following chain of transformations

$$F_\mu(s,x) = \sum_{m=0}^{\infty} \frac{s^m}{m!} \left(\sum_{l=0}^{m} S_\mu(m,l) x^l \right) =$$

$$\sum_{m=0}^{\infty} \frac{s^m}{m!} \left(\sum_{l=0}^{m} \theta(m-l) S_\mu(m,l) x^l \right) = \sum_{l=0}^{\infty} \left(\sum_{m=l}^{\infty} S_\mu(m,l) \frac{s^m}{m!} \right) x^l, \tag{67}$$

where $\theta(m-l)$ is the Heaviside step function defined by Eq.(58).

On the other hand, from Eq.(45), we have for $F_\mu(s,x)$

$$F_\mu(s,x) = \sum_{l=0}^{\infty} \frac{(e^s-1)^l}{\Gamma(\mu l+1)} x^l. \tag{68}$$

Upon comparing this equation and Eq.(67), we conclude that

$$\sum_{m=l}^{\infty} S_\mu(m,l) \frac{s^m}{m!} = \frac{(e^s-1)^l}{\Gamma(\mu l+1)}, \qquad l = 0,1,2,.... \tag{69}$$

Now we are set up to define two generating functions $\mathcal{G}_\mu(s,l)$ and $\mathcal{F}_\mu(s,t)$ of the fractional Stirling numbers of the second kind,

$$\mathcal{G}_\mu(s,l) = \sum_{m=l}^{\infty} S_\mu(m,l)\frac{s^m}{m!} = \frac{(e^s-1)^l}{\Gamma(\mu l+1)}, \tag{70}$$

$$\mathcal{F}_\mu(s,t) = \sum_{m=0}^{\infty}\sum_{l=0}^{m} S_\mu(m,l)\frac{s^m t^l}{m!} = \sum_{l=0}^{\infty}\frac{t^l(e^s-1)^l}{\Gamma(\mu l+1)} = E_\mu(t(e^s-1)). \tag{71}$$

As a special case $\mu = 1$, equations (70) and (71) include the well-know generating function equations for the standard Stirling numbers of the second kind $S(m,l)$ (for instance, see Eqs.(2.17) and (2.18) in Ref.[14]),

$$\mathcal{G}_1(s,l) = \mathcal{G}_\mu(s,l)\mid_{\mu=1} = \sum_{m=l}^{\infty} S(m,l)\frac{s^m}{m!} = \frac{(e^s-1)^l}{l!}, \qquad l = 0,1,2,.... \tag{72}$$

and

$$\mathcal{F}_1(s,l) = \mathcal{F}_\mu(s,l)\mid_{\mu=1} = \sum_{m=0}^{\infty}\sum_{l=0}^{m} S(m,l)\frac{s^m t^l}{m!} = \exp(t(e^s-1)), \quad l = 0,1,2,.... \tag{73}$$

4.2.1 The Bernoulli numbers and fractional Stirling numbers of the second kind

The Bernoulli numbers B_n, $n = 0,1,2,...$have the generating function [28],

$$\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = \frac{t}{e^t-1}. \tag{74}$$

These numbers play an important role in the number theory.

Let's show that the Bernoulli numbers B_n can be presented in terms of the fractional Stirling numbers of the second kind $S_\mu(n,k)$ in the following way

$$B_n = \sum_{k=0}^{n}(-1)^k\Gamma(\mu k+1)\frac{S_\mu(n,k)}{k+1}. \tag{75}$$

To prove Eq.(75) we substitute B_n from Eq.(75) into the left hand side of Eq.(74). Therefore, we have

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\Gamma(\mu k+1)\frac{S_\mu(n,k)}{k+1}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k\Gamma(\mu k+1)\frac{S_\mu(n,k)}{k+1}\frac{t^n}{n!}. \tag{76}$$

To transform the right hand side of Eq.(76) we use Eq.(70) and obtain

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k\Gamma(\mu k+1)\frac{S_\mu(n,k)}{k+1}\frac{t^n}{n!} = \sum_{k=0}^{\infty}(-1)^k\frac{(e^t-1)^k}{k+1} = \frac{t}{e^t-1}.$$

Thus, we have proved Eq.(75).

In the case $\mu = 1$, when the gamma function is $\Gamma(\mu k+1)|_{\mu=1} = k!$, Eq.(75) reads

$$B_n = \sum_{k=0}^{n}(-1)^k k!\frac{S(n,k)}{k+1}, \tag{77}$$

and we recover the representation of the Bernoulli numbers B_n in terms of the Stirling numbers of the second kind $S(n,k)$ (for instance, see the equation for B_n on page 2547 of Ref.[14]).

5 Statistics of the fractional Poisson probability distribution

Now we use the fractional Stirling numbers introduced by Eq.(60) to get the moments and the central moments of the fractional Poisson process with the probability distribution function $P_\mu(n,t)$ given by Eq.(7). Indeed, by definition of the m-th order moment of the fractional Poisson probability distribution we have

$$\overline{n_\mu^m} = \sum_{n=0}^{\infty} n^m P_\mu(n,t) = \sum_{n=0}^{\infty} n^m \frac{(\nu t^\mu)^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1)}, \quad 0 < \mu \le 1. \tag{78}$$

It is easy to see that $\overline{n^m_\mu}$ is in fact the fractional Bell polynomial $B_\mu(\nu t^\mu, m)$ defined by Eq.(37).

From other side, with the help of Eqs.(37) and (55) we find

$$\overline{n^m_\mu} = \sum_{l=0}^{m} S_\mu(m,l)(\nu t^\mu)^l. \tag{79}$$

Hence, the fractional Stirling numbers $S_\mu(m,l)$ of the second kind naturally appear in the power series of the m-th order moment of the fractional Poisson probability distribution.

Using analytical expressions given by Eqs.(60) and (79), let's list a few moments of the fractional Poisson probability distribution

$$\overline{n}_\mu = \sum_{n=0}^{\infty} nP_\mu(n,t) = \frac{\nu t^\mu}{\Gamma(\mu+1)}, \tag{80}$$

$$\overline{n^2_\mu} = \sum_{n=0}^{\infty} n^2 P_\mu(n,t) = \frac{2(\nu t^\mu)^2}{\Gamma(2\mu+1)} + \frac{\nu t^\mu}{\Gamma(\mu+1)}, \tag{81}$$

$$\overline{n^3_\mu} = \sum_{n=0}^{\infty} n^3 P_\mu(n,t) = \frac{6(\nu t^\mu)^3}{\Gamma(3\mu+1)} + \frac{6(\nu t^\mu)^2}{\Gamma(2\mu+1)} + \frac{\nu t^\mu}{\Gamma(\mu+1)}, \tag{82}$$

$$\overline{n^4_\mu} = \sum_{n=0}^{\infty} n^4 P_\mu(n,t) = \frac{24(\nu t^\mu)^4}{\Gamma(4\mu+1)} + \frac{36(\nu t^\mu)^3}{\Gamma(3\mu+1)} + \frac{14(\nu t^\mu)^2}{\Gamma(2\mu+1)} + \frac{\nu t^\mu}{\Gamma(\mu+1)}. \tag{83}$$

In terms of the power series over the first order moment $\overline{n}_\mu$, the above equations (81) - (83) read

$$\overline{n^2_\mu} = \frac{2(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}\overline{n}^2_\mu + \overline{n}_\mu, \tag{84}$$

$$\overline{n^3_\mu} = \frac{6(\Gamma(\mu+1))^3}{\Gamma(3\mu+1)}\overline{n}^3_\mu + \frac{6(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}\overline{n}^2_\mu + \overline{n}_\mu, \tag{85}$$

$$\overline{n^4_\mu} = \frac{24(\Gamma(\mu+1))^4}{\Gamma(4\mu+1)}\overline{n}^4_\mu + \frac{36(\Gamma(\mu+1))^3}{\Gamma(3\mu+1)}\overline{n}^3_\mu + \frac{14(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}\overline{n}^2_\mu + \overline{n}_\mu. \tag{86}$$

The mean $\overline{n}_\mu$ Eq.(80) and the second order moment $\overline{n_\mu^2}$ were first obtained by Laskin[5] (see, Eqs.(26) and (27) in Ref.[5]).

In the case when $\mu = 1$, equations (84) - (86) become the well-know equations for moments of the standard Poisson probability distribution with the parameter $\overline{n} \equiv \overline{n}_1 = \nu t$ (for instance, see Eqs.(22) - (24) in Ref. [29]).

5.1 Variance, skewness and kurtosis of the fractional Poisson probability distribution

To find analytical expressions for variance, skewness and kurtosis of the fractional Poisson probability distribution, let's introduce the central m-th order moment $M_\mu(m)$

$$M_\mu(m) = \overline{(n_\mu - \overline{n}_\mu)^m} = \sum_{n=0}^{\infty}(n-\overline{n}_\mu)^m P_\mu(n,t) =$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r} n^r(\overline{n}_\mu)^{m-r}P_\mu(n,t) = \tag{87}$$

$$\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}(\overline{n}_\mu)^{m-r}\sum_{l=0}^{r}S_\mu(r,l)(\nu t^\mu)^l,$$

where $S_\mu(r,l)$ is given by Eq.(60).

Hence, in terms of power series over the first order moment Eq.(80), we have

$$M_\mu(1) = 0, \tag{88}$$

$$M_\mu(2) = \left(\frac{2(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)} - 1\right)\overline{n}_\mu^2 + \overline{n}_\mu, \tag{89}$$

$$M_\mu(3) = 2\left(\frac{3(\Gamma(\mu+1))^3}{\Gamma(3\mu+1)} - \frac{3(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)} + 1\right)\overline{n}_\mu^3 + \tag{90}$$

[5]The second order moment defined by Eq.(81) can be presented as Eq.(27) of Ref.[5] if we take into account the well-know equations for the gamma function $\Gamma(\mu)$

$$\Gamma(\mu+1) = \mu\Gamma(\mu), \qquad \Gamma(2\mu) = \frac{2^{2\mu-1}}{\sqrt{\pi}}\Gamma(\mu)\cdot\Gamma(\mu+\frac{1}{2}).$$

$$3\left(\frac{2(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}-1\right)\overline{n}_\mu^2+\overline{n}_\mu,$$

$$M_\mu(4)=3\left(\frac{8(\Gamma(\mu+1))^4}{\Gamma(4\mu+1)}-\frac{8(\Gamma(\mu+1))^3}{\Gamma(3\mu+1)}+\frac{4(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}-1\right)\overline{n}_\mu^4+ \quad (91)$$

$$6\left(\frac{6(\Gamma(\mu+1))^3}{\Gamma(3\mu+1)}-\frac{4(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}+1\right)\overline{n}_\mu^3+2\left(\frac{7(\Gamma(\mu+1))^2}{\Gamma(2\mu+1)}-2\right)\overline{n}_\mu^2+\overline{n}_\mu.$$

Further, in terms of the above defined central moments $M_\mu(m)$, the variance σ^2, skewness s_μ, and kurtosis k_μ of the fractional Poisson probability distribution are

$$\sigma_\mu^2=M_\mu(2), \quad (92)$$

$$s_\mu=\frac{M_\mu(3)}{M_\mu^{3/2}(2)}, \quad (93)$$

$$k_\mu=\frac{M_\mu(4)}{M_\mu^2(2)}-3. \quad (94)$$

In the case when $\mu=1$, new equations (89) - (91) turn into equations for the central moments of the standard Poisson probability distribution (see, Eqs.(25) - (27) in Ref. [29]).

Equations (92) - (94), at $\mu=1$, turn into the equations for variance, skewness, and kurtosis of the standard Poisson probability distribution with the parameter $\overline{n}$ (for instance, see Eqs.(29) - (31) in Ref. [29]).

Table 4 summarizes equations for the Bell polynomials $B(x,m)$, the Bell numbers $B(m)$, generating function of the Bell numbers $\mathcal{B}(s)$, the Stirling numbers of the second kind $S(m,l)$, generating function of the Stirling numbers of the second kind $\sum\limits_{m=l}^{\infty}S_\mu(m,l)\frac{s^m}{m!}$, and the m-th order moment, attributed to the fractional Poisson distribution with those for the standard Poisson distribution.

	fractional Poisson ($0 < \mu \leq 1$)	Poisson ($\mu = 1$)
$B(x,m)$	$\sum\limits_{n=0}^{\infty} n^m \frac{x^n}{n!} \sum\limits_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-x)^k}{\Gamma(\mu(k+n)+1))}$	$e^{-x} \sum\limits_{n=0}^{\infty} n^m \frac{x^n}{n!}$
$B(m)$	$\sum\limits_{n=0}^{\infty} (-1)^n \frac{n^m}{n!} E_\mu^{(n)}(-1)$	$e^{-x} \sum\limits_{n=0}^{\infty} \frac{n^m}{n!}$
$\mathcal{B}(s)$	$E_\mu(e^s - 1)$	$\exp\{(e^s - 1)\}$
$S(m,l)$	$\frac{1}{\Gamma(\mu l+1)} \sum\limits_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m$	$\frac{1}{l!} \sum\limits_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m$
$\sum\limits_{m=l}^{\infty} S_\mu(m,l) \frac{s^m}{m!}$	$\frac{(e^s-1)^l}{\Gamma(\mu l+1)}$	$\frac{(e^s-1)^l}{l!}$
$\overline{n^m}$	$\sum\limits_{l=0}^{m} S_\mu(m,l)(\nu t^\mu)^l$	$\sum\limits_{l=0}^{m} S(m,l)(\nu t)^l$

Table 4. *Polynomials, numbers, moments and generating functions attributed to the fractional Poisson process vs the standard Poisson process.*

6 Conclusions

The quantum physics and the number theory applications of the fractional Poisson probability distribution have been developed.

The key new results related to quantum physics applications are given by Eqs.(14), (15), (17), (25), (31) - (36).

The key new results related to the number theory applications are given by Eqs.(37), (40), (45), (47), (55), (60), (70), (71), (75), (79), (80) - (83), (87) - (91).

Tables 2, 3, 4 summarize our findings for the fractional Poisson probability distribution in comparison to the well-known results related to the standard Poisson probability distribution.

These findings facilitate the further exploration of other fields to apply the fractional Poisson probability distribution and will provide deep insight into long-memory impacts on counting processes.

7 Appendix

To obtain Eq.(24) we use the Laplace transform of the Mittag-Leffler function $E_\mu(-z\tau^\mu)$

$$\int_0^\infty d\tau e^{-\tau} E_\mu(-z\tau^\mu) = \frac{1}{1+z},$$

for instance, see equation (26) on the page 210 in Ref.[25].
Changing the variable $\tau \to st$ and the parameter $zs^\mu \to \zeta$ yields

$$\int_0^\infty dt e^{-st} E_\mu(-\zeta t^\mu) = \frac{s^{\mu-1}}{s^\mu + \zeta}. \tag{95}$$

By differentiating Eq.(95) n times with respect to ζ we obtain,

$$\int_0^\infty dt e^{-st} t^{\mu n} E_\mu^{(n)}(-\zeta t^\mu) = \frac{n! \cdot s^{\mu-1}}{(s^\mu + \zeta)^{n+1}}. \tag{96}$$

It is easy to see that at $\zeta = \mu$ Eq.(96) goes into Eq.(24).

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