ELSEVIER

Physica A 287 (2000) 482–492

PHYSICA A

www.elsevier.com/locate/physa

Fractional market dynamics

Nick Laskin

Department of Systems and Computer Engineering, Carleton University, 1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada

Received 30 April 2000; received in revised form 12 June 2000

Abstract

A new extension of a fractality concept in financial mathematics has been developed. We have introduced a new fractional Langevin-type stochastic differential equation that differs from the standard Langevin equation: (i) by replacing the first-order derivative with respect to time by the fractional derivative of order μ; and (ii) by replacing "white noise" Gaussian stochastic force by the generalized "shot noise", each pulse of which has a random amplitude with the α-stable Lévy distribution. As an application of the developed fractional non-Gaussian dynamical approach the expression for the probability distribution function (pdf) of the returns has been established. It is shown that the obtained fractional pdf fits well the central part and the tails of the empirical distribution of S&P 500 returns.

PACS: 02.50.Ey; 05.45.Df; 05.40.Fb

Keywords: Fractal; Shot noise; Lévy α-stable process; Return

1. Introduction

Dynamics of financial assets demonstrate the stochastic behavior. The first theoretical attempt to describe stochastic financial dynamics was made by Bachelier in 1900 [1]. He proposed the Brownian motion to model the stochastic process of the return $G(t) \equiv G_{\Delta t}(t)$ over a time scale Δt defined as the forward change in the logarithm of price or market index $S(t)$,

$$G_{\Delta t}(t) = \ln S(t + \Delta t) - \ln S(t) .$$

Bachelier's approach is natural if one considers the return over a time scale Δt to be the result of many independent "shocks", which then lead by the central limit theorem to a Gaussian distribution of returns [1]. The Gaussian assumption for the

E-mail address: nlaskin@sce.carleton.ca, nlaskin@rocketmail.com (N. Laskin).

0378-4371/00/$ - see front matter
PII: S0378-4371(00)00387-3

dynamics of a financial assets is widely used in mathematical finance because of the simplifications it provides in analytical calculation; indeed, it is the main assumption used in the famous Black–Scholes option pricing formula [2].

However, empirical studies [3–5] showed that the probability distribution of returns has pronounced tails in striking contrast to that of a Gaussian.

Mandelbrot, who introduced into scientists' lexicon the new term "fractal", observed [3] that in addition to being non-Gaussian, the stochastic process of returns shows another interesting property: self-similarity – that is, the statistical dependencies of returns have similar form for various time increments Δt, ranging from 1 d to 1 month. As it is discussed in Ref. [6] "motivated by (i) the pronounced tails, and (ii) the stable functional form for different time scales, Mandelbrot [3] proposed that the distributions of the returns is consistent with a Lévy stable distribution [7] – that is, the returns can be modeled as a Lévy α-stable process". Thus, from the point of view of the fractal concept one may say that Bachelier's and Mandelbrot's approaches were the first attempts applying the fractality concept to model the financial assets dynamics. It is well known that the trajectories of the Brownian and Lévy stochastic processes are fractals. It means that they are non-differentiable, self-similar curves whose fractal dimensions are different from their topological dimension [8].

Since the well-known papers [3,9] on Lévy distributions, there have been several attempts to develop the fractional approach to the problem. Most of them deal with cut-off of the Lévy distributions (see, for example, Refs. [10,11]). The approaches based on cut-off procedures are approximations to the pdf trying to fit the empirical data, but they are essentially non-dynamical and do not allow one to predict the future behavior of a market.

We develop a new extension of a fractality concept in financial mathematics and apply it to describe the stochastic dynamics of the stock and currency markets. We propose a new fractional dynamical approach to model the evolution of the stochastic financial assets. The main difference from the previous stochastic dynamical approaches to fluctuating market phenomena is the following. We consider the fractional Langevin-type stochastic differential equation that differs from the standard Langevin equation:

(i) By replacing the first derivative with respect to time by the fractional derivative of order μ.

(ii) By replacing the "white noise" Gaussian stochastic force by the generalized "shot noise", each pulse of which has a random amplitude.

The proposed fractional dynamical stochastic approach allows to obtain the probability distribution function (pdf) of the modeled financial asset. As an application of the developed general approach, we derive the equation for the pdf of increments Δx of a financial market index as a function of the time delay Δt, $\Delta x(\Delta t) = x(t + \Delta t) - x(t)$, where the value of the index is denoted as $x(t)$. Statistical properties of asset price increments play an important role both for understanding of the markets dynamics and for financial engineering applications, for instance, the pricing of derivative products and risk evaluations. The theoretically predicted pdf of increments of market index Δx

as a function of the time delay Δt has been compared with the well-known statistical dependencies of the Standard&Poor's 500 index (S&P 500). It is shown that the developed fractional pdf fits well as the central part, as the tails of the distribution of S&P 500.

The main goal of the fitting is to calibrate the numerical parameters of the proposed fractional dynamical stochastic model for the considered market. After that, the calibrated dynamical model allows one to predict statistically the future dynamical behavior of this market.

The paper is organized as follows.

In Section 2 we describe evolution of a financial asset by the fractional Langevin equation with the generalized shot noise source. The pdf of the financial asset has been expressed in terms of the Fourier integral.

The stochastic fractional dynamics of the variation of a financial market index (a return) has been developed in Section 3. The pdf of returns Δx as a function of the time delay Δt has been found and some limiting cases are studied.

The results of numerical simulations of the developed general equation for the pdf of returns are presented in Section 4. These results are compared with the empirical data of the S&P 500 index for the 6-year period from January 1984 to December 1989 [4]. The numerical analysis confirms that the developed fractional pdf fits well the central part and the asymptotic tail's behavior of the S&P 500 index distribution.

In the conclusion we discuss the fractional nature of the market dynamics.

2. Fractional stochastic dynamic model

We propose to describe the dynamics of a financial asset $x(t)$ by the fractional stochastic differential equation

$$\frac{d^{\mu}x(t)}{dt^{\mu}} = \lambda x(t) + F(t), \quad 0 < \mu \leqslant 1 , \tag{1}$$

with initial condition

$$\left.\frac{d^{\mu-1}x(t)}{dt^{\mu-1}}\right|_{t=0} = x_0 , \tag{2}$$

where λ is the expected rate, $F(t)$ is the random force and d^{μ}/dt^{μ} means the Riemann–Liouville fractional derivative [1] of order μ,

$$\frac{d^{\mu}x(t)}{dt^{\mu}} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_0^t dt' \frac{x(t')}{(t-t')^{\mu}} .$$

Using the definition of the Riemann–Liouville fractional integral [12,13], $d^{-\mu}/dt^{-\mu}$ of the function $x(t)$,

$$\frac{d^{-\mu}x(t)}{dt^{-\mu}} \equiv {}_0 I_t^{\mu}(x) = \frac{1}{\Gamma(\mu)}\int_0^t dt' \frac{x(t')}{(t-t')^{1-\mu}}, \quad 0 < \mu \leqslant 1$$

[1] The basic formulas on fractional calculus can be found in Refs. [12,13].

yields

$$x(t;x_0,F)=t^{\mu-1}E_{\mu,\mu}(\lambda t^{\mu})x_0+\int_0^t \mathrm{d}\tau\,F(\tau)(t-\tau)^{\mu-1}E_{\mu,\mu}(\lambda(t-\tau)^{\mu})\,. \tag{3}$$

Here the gamma function $\Gamma(z)$ is defined as $\Gamma(z)=\int_0^{\infty}\mathrm{d}t t^{z-1}\mathrm{e}^{-t}$, $\mathrm{Re}\,z>0$ and $E_{\mu,\rho}(z)$ is the so-called generalized Mittag–Leffler function defined by the series (see, for instance, Ref. [14])

$$E_{\mu,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\rho)}\,. \tag{4}$$

The fractality index μ is related to Mandelbrot's [8,15] self-similarity parameter H as

$$\mu=H+\tfrac{1}{2}\,.$$

The mathematical motivation for applying the fractional stochastic problem (1), (2) is the following. It is easy to see when $\mu=1$, Eq. (1) reduces to the standard (non-fractional) Langevin equation with initial condition

$$x(t)|_{t=0}=x_0\,,$$

and Eq. (3) gives the solution of this standard, the well-known stochastic problem

$$x(t;x_0,F)=\mathrm{e}^{\lambda t}x_0+\int_0^t \mathrm{d}\tau\,F(\tau)\mathrm{e}^{\lambda(t-\tau)}$$

because of

$$E_{1,1}(z)=\mathrm{e}^{z}\,.$$

Thus, we see that the fractional stochastic initial problem (1), (2) seems as a fractional generalization of the well-known Langevin approach to fluctuating phenomena.

We define the probability distribution function $P_{\mu}(x,t)$ of the fractional stochastic variable $x(t)$ in the following way:

$$\begin{aligned}P_{\mu}(x,t)&=\langle\delta(x-x(t;x_0,F))\rangle\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\exp\{\mathrm{i}\xi(x-t^{\mu-1}E_{\mu,\mu}(\lambda t^{\mu})x_0)\}\\&\quad\times\left\langle\exp\left\{-\mathrm{i}\xi\int_0^t \mathrm{d}\tau\,F(\tau)(t-\tau)^{\mu-1}E_{\mu,\mu}(\lambda(t-\tau)^{\mu})\right\}\right\rangle\,,\end{aligned} \tag{5}$$

where the brackets $\langle\cdots\rangle$ mean the averaging over the all possible realizations of the random force $F(t)$.

Let the stochastic force $F(t)$ be a generalized shot noise

$$F(t)=\sum_{k=1}^{n}a_k\varphi(t-t_k)\,. \tag{6}$$

Here a_k are the random amplitudes, $\varphi(t)$ is the response (or memory) function, and t_k are the homogeneously distributed (on time interval $[0,T]$) moments of time, the number n of which obeys the Poisson law.

We guess that, defined by Eq. (6), random force $F(t)$ describes the influence of the different fluctuating factors on the market dynamics. A single-shot-noise pulse $a_k\varphi(t-t_k)$ describes the influence of a piece of information which has become available at the random moment t_k on the decision-making process at a later time t. The amplitude a_k responds to the magnitude of the pulse $\varphi(t-t_k)$; it will depend on the type of information and will, therefore, be subjected to probability distribution. For simplicity, we assume that each pulse has the same functional form or, in other words, one general response function φ can be used to describe the market.

Thus, the averaging procedure includes three statistically independent averaging procedures:

1. Averaging over random amplitudes a_k, $\langle\cdots\rangle_{a_k}$,
$$\langle\cdots\rangle_{a_k} = \int \mathrm{d}a_1\cdots\mathrm{d}a_n\, P(a_1,\ldots,a_n)\cdots\ , \tag{7}$$
where $P(a_1,\ldots,a_n)$ is the probability distribution of amplitudes a_k.
2. Averaging over t_k on time interval T,
$$\langle\cdots\rangle_T = \frac{1}{T}\int_0^T \mathrm{d}t_1\cdots\frac{1}{T}\int_0^T \mathrm{d}t_n\cdots\ . \tag{8}$$
3. Averaging over random numbers n of time moments t_k,
$$\langle\cdots\rangle_n = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\mathrm{e}^{-\bar{n}}\cdots\ , \tag{9}$$
where $\bar{n} = vT$ and v is the density of points t_k on time interval T.

Taking into account the definition, Eq. (5), and performing the averaging in accordance with Eqs. (7)–(9) yields
$$P_\mu(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\exp\{\mathrm{i}\xi(x - t^{\mu-1}E_{\mu,\mu}(\lambda t^\mu)x_0)\}\exp\{-J_\mu(\xi,t)\}\ , \tag{10}$$
where we introduce the following notation:
$$J_\mu(\xi,t) = v\int_0^t \mathrm{d}t'\left[1 - W\left(\xi\int_0^t \mathrm{d}\tau\,\varphi(\tau-t')(t-\tau)^{\mu-1}E_{\mu,\mu}(\lambda(t-\tau)^\mu)\right)\right]\ . \tag{11}$$
Here the function $W(\varsigma)$ is the characteristic function of the probability distribution $P_1(a)$,
$$W(\varsigma) = \int_{-\infty}^{\infty}\mathrm{d}a\,\mathrm{e}^{-\mathrm{i}\varsigma a}P_1(a)$$
and the pdf $P_1(a)$ is a “one-particle” distribution to be introduced into consideration because of simplicity assumption
$$P(a_1,\ldots,a_n) = \prod_{k=1}^{n}P_1(a_k)\ . \tag{12}$$

In other words, we consider a market dynamics when probability distribution $P(a_1,\ldots,a_n)$ is factorized in accordance with Eq. (12) as the product of n equal "one-particle" distributions $P_1(a)$.

To evaluate the integrals in Eqs. (10) and (11) we should specify the response function $\varphi(t)$ and the pdf $P_1(a)$. As a first step, let us choose an exponential response function

$$\varphi(t) = \begin{cases} \exp\{-\frac{t}{\delta}\}, & t \geqslant 0\,, \\ 0, & t < 0\,, \end{cases} \tag{13}$$

which means that the impact has a characteristic memory time δ, and evaluate the integral over $\mathrm{d}\tau$ in Eq. (11),

$$\int_0^t \mathrm{d}\tau\, \varphi(\tau - t')(t-\tau)^{\mu-1} E_{\mu,\mu}(\lambda(t-\tau)^\mu) = \int_{t'}^t \mathrm{d}\tau\, \mathrm{e}^{-(\tau-t')/\delta}(t-\tau)^{\mu-1} \sum_{k=0}^{\infty} \frac{\lambda^k (t-\tau)^{\mu k}}{\Gamma(\mu k + \mu)}\,. \tag{14}$$

Expanding in series the $\mathrm{e}^{-(\tau-t')/\delta}$ and using the formula

$$\int_{t'}^t \mathrm{d}\tau\, (t-\tau)^{a-1}(\tau - t')^{b-1} = (t-t')^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,,$$

gives for the right-hand side of Eq. (14)

$$\int_{t'}^t \mathrm{d}\tau\, \mathrm{e}^{-(\tau-t')/\delta}(t-\tau)^{\mu-1} \sum_{k=0}^{\infty} \frac{\lambda^k (t-\tau)^{\mu k}}{\Gamma(\mu k + \mu)} = (t-t')^\mu \sum_{k=0}^{\infty} \left(-\frac{t-t'}{\delta}\right)^k E_{\mu,\mu+k+1}(\lambda(t-t')^\mu)\,,$$

where $E_{\mu,\mu+k+1}$ is the generalized Mittag–Leffler function defined by Eq. (4). The function $J_\mu(\xi,t)$ given by Eq. (11) then reads

$$J_\mu(\xi,t) = v \int_0^t \mathrm{d}\tau\, [1 - W(\xi \mathscr{R}_\mu(\tau;\lambda))]\,, \tag{15}$$

with

$$\mathscr{R}_\mu(\tau;\lambda) = \tau^\mu \sum_{k=0}^{\infty} \left(-\frac{\tau}{\delta}\right)^k E_{\mu,\mu+k+1}(\lambda\tau^\mu)\,. \tag{16}$$

As a second step, let us choose the Lévy α-stable distribution $P_1(a)$ as a "one-particle" probability distribution function

$$P_1(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\varsigma \mathrm{e}^{\mathrm{i}\varsigma a} W(\varsigma)\,,$$

with the characteristic function $W(\varsigma)$,

$$W(\varsigma) = \exp\{-b^\alpha |\varsigma|^\alpha\}, \quad 0 < \alpha \leqslant 2\,, \tag{17}$$

where b is the scale parameter of the Lévy α-stable distribution.

Thus, in accordance with Eq. (10) the new general equation for the pdf of the fractional stochastic process $x(t)$ modelled by Eq. (1) can be rewritten as

$$P_\mu(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \exp\{\mathrm{i}\xi(x - t^{\mu-1}E_{\mu,\mu}(\lambda t^\mu)x_0)\} \times \exp\left\{-v\int_0^t \mathrm{d}\tau\,[1-\exp\{-b^\alpha|\xi\mathcal{R}_\mu(\tau;\lambda)|^\alpha\}]\right\}, \tag{18}$$

with $\mathcal{R}_\mu(\tau)$ defined by Eq. (16).[2]

In order to verify the adequacy of the developed fractional model, we should compare the predicted theoretical result with the empirical financial data. In Section 3 we apply the general equation (18) and study the well-known statistical dependencies of the Standard&Poor's 500 index (S&P 500).

3. Fractional returns dynamics

Let us apply the developed general approach to derive the analytical expression for the pdf of increments Δx of a financial market index as a function of the time delay Δt, $\Delta x = x(t+\Delta t) - x(t)$, where the value of the market index is denoted as $x(t)$. The pdf of price increments fluctuations plays an important role both in understanding the market dynamics and in financial engineering applications.

As usual, we define the pdf $P_\mu(\Delta x, t, \Delta t;\alpha)$ of the increments $\Delta x(\Delta t) = x(t+\Delta t) - x(t)$ of a financial market index $x(t)$ (for example, a currency rate) as a function of the time delay Δt by the following equality:

$$P_{\alpha,\mu}(\Delta x, t, \Delta t) = \langle \delta(\Delta x - \{x(t+\Delta t; x_0, F) - x(t; x_0, F)\})\rangle\,, \tag{19}$$

where $\langle\cdots\rangle$ means averaging over the all possible realizations of the random force $F(t)$ in accordance with Eqs. (7)–(9). Repeating the same steps used above for derivation of Eq. (18), we find for the pdf $P_{\alpha,\mu}(\Delta x, t, \Delta t)$

$$P_{\alpha,\mu}(\Delta x, t, \Delta t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\xi \cos\xi\{\Delta x - ((t+\Delta t)^{\mu-1}E_{\mu,\mu}(\lambda(t+\Delta t)^\mu) - t^{\mu-1}E_{\mu,\mu}(\lambda t^\mu))x_0\}\exp\{-L_{\alpha,\mu}(\xi, t, \Delta t;\lambda)\}\,, \tag{20}$$

with

$$L_{\alpha,\mu}(\xi, t, \Delta t;\lambda) = v\int_0^t \mathrm{d}\tau\,(1 - \exp\{-b^\alpha\xi^\alpha|\mathcal{R}_\mu(\tau+\Delta t;\lambda) - \mathcal{R}_\mu(\tau;\lambda)|^\alpha\})\,,$$

where $\mathcal{R}_\mu(\tau;\lambda)$ is given by Eq. (16).

[2] Note that if we put $\mu = 1$, then Eq. (16) yeilds

$$\mathcal{R}_1(\tau) = (\delta/\lambda\delta + 1)(\mathrm{e}^{\lambda\tau} - \mathrm{e}^{-\tau/\delta})\,.$$

Eq. (20) presents the new general expression for the fractional pdf of price increments $\Delta x = x(t+\Delta t) - x(t)$ fluctuations, when the price $x(t)$ dynamics is described by the fractional stochastic differential equation (1).

In order to escape unwieldy formulas and in view of the fact that our goal is only to illustrate the developed fractional general approach, we restrict our consideration to the case $\lambda = 0$. In this case, Eq. (20) can be rewritten as

$$\begin{aligned} P_{\alpha,\mu}(\Delta x, t, \Delta t) \\ = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}\xi \cos\xi \left\{\Delta x - \frac{1}{\Gamma(\mu)}((t+\Delta t)^{\mu-1} - t^{\mu-1})x_0\right\} \\ \times \exp\{-L_{\alpha,\mu}(\xi, t, \Delta t)\}\,, \end{aligned} \tag{21}$$

where $L_{\alpha,\mu}(\xi, t, \Delta t)$ has a form

$$L_{\alpha,\mu}(\xi, t, \Delta t) = v\int_0^{t} \mathrm{d}\tau\,(1 - \exp\{-b^{\alpha}\xi^{\alpha}\cdot|r_{\mu}(\tau+\Delta t) - r_{\mu}(\tau)|^{\alpha})$$

and $r_{\mu}(\tau)$ is obtained from $\mathscr{R}_{\mu}(\tau;\lambda)$ (see Eq. (16)) by passing to $\lambda = 0$,

$$r_{\mu}(\tau) \equiv \mathscr{R}_{\mu}(\tau;\lambda=0) = \tau^{\mu}E_{1,1+\mu}\left(-\frac{\tau}{\delta}\right)\,. \tag{22}$$

Further, we will be interested in the limit case $t \to \infty$, when we have

$$P_{\alpha,\mu}(\Delta x, \Delta t) = \lim_{t\to\infty} P_{\alpha,\mu}(\Delta x, t, \Delta t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}\xi \cos(\xi\,\Delta x) \times \mathrm{e}^{-L_{\alpha,\mu}(\xi,\Delta t)}\,. \tag{23}$$

Here $L_{\alpha,\mu}(\xi, \Delta t)$ is defined by

$$\begin{aligned} L_{\alpha,\mu}(\xi, \Delta t) = \lim_{t\to\infty} L_{\alpha,\mu}(\xi, t, \Delta t) \\ = v\int_0^{\infty} \mathrm{d}\tau\,(1 - \exp\{-b^{\alpha}\xi^{\alpha}\cdot|r_{\mu}(\tau+\Delta t) - r_{\mu}(\tau)|^{\alpha}\})\,, \\ 0 < \mu \leqslant 1,\quad 1 < \alpha \leqslant 2\,. \end{aligned} \tag{24}$$

The limiting pdf $P_{\alpha,\mu}(\Delta x, \Delta t)$ is characterized by the fractality index μ and the Lévy index α. Thus, it is shown how the general fractional dynamic approach developed in Section 2 allows one to derive the expression (see Eqs. (23) and (24)) for the pdf $P_{\alpha,\mu}(\Delta x, \Delta t)$ of the returns. The new pdf $P_{\alpha,\mu}(\Delta x, \Delta t)$ allows to study any statistical and scaling dependences of the returns fluctuating dynamics and develop the new general fractional approach to risk evaluations and the pricing of derivative products.

In the special (non-fractional) case when $\mu = 1$ the general Eq. (24) can be represented as

$$L_{\alpha,1}(\xi, \Delta t) = v\int_0^{\infty} \mathrm{d}\tau\,\{1 - \exp[-b^{\alpha}|\xi|^{\alpha}\delta^{\alpha}|\mathrm{e}^{-\tau/\delta}(1 - \mathrm{e}^{-\Delta t/\delta})|^{\alpha}]\}\,, \tag{25}$$

where we have kept in view that $\Gamma(1)=1$ and

$$E_{1,2}(z)=\frac{e^z-1}{z}\,.$$

Then Eq. (23) leads to

$$P_{\alpha,1}(\Delta x,\Delta t)=\frac{1}{\pi}\int_0^\infty d\xi\cos(\xi\Delta x)$$
$$\times\exp\left\{-v\int_0^\infty d\tau\,\{1-\exp[-b^\alpha\xi^\alpha\delta^\alpha(e^{-\tau/\delta}(1-e^{-\Delta t/\delta}))^\alpha]\}\right.\,.$$

Introducing the new variable u instead of τ

$$u=b\xi\delta e^{-\tau/\delta}(1-e^{-\Delta t/\delta}),\quad du=-\frac{u}{\delta}\,d\tau\,,$$

we get the expression for $P_{\alpha,1}(\Delta x,\Delta t)$,

$$P_{\alpha,1}(\Delta x,\Delta t)=\frac{1}{\pi}\int_0^\infty d\xi\cos(\xi\Delta x)\exp\left\{-v\delta\int_0^{b\delta(1-e^{-\Delta t/\delta})\xi}\frac{du}{u}(1-e^{-u^\alpha})\right\}\,. \tag{26}$$

The pdf $P_{\alpha,1}(\Delta x,\Delta t)$ is the α-generalization of Eqs. (8) and (9) of Ref. [16]. In the Gaussian case ($\alpha=2$) the pdf $P_{2,1}(\Delta x,\Delta t)$ was obtained in Ref. [16].

4. Comparison with empirical data

The comparison of the developed fractional pdf with the well-known empirical pdf of the S&P 500 returns [4] is represented in Fig. 1. The circles represent the empirical data obtained from the Chicago Mercantile Exchange during the period from January 1984 to December 1989. The variations of S&P 500 index are normalized to the value

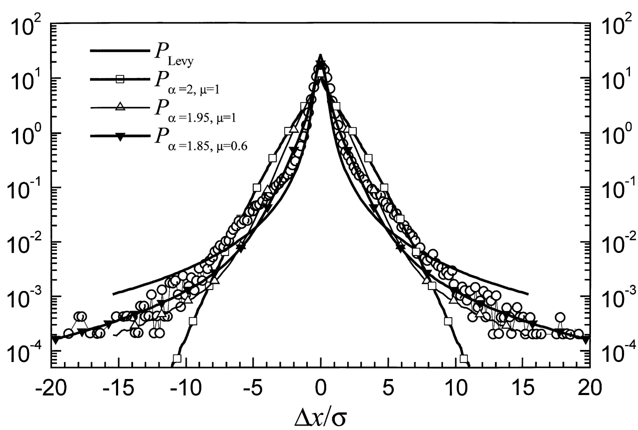

Fig. 1. Comparison of the S&P 500 data, the Lévy distribution, and the fractional pdf $P_{\alpha,\mu}(\Delta x/\sigma,\Delta\tau=0.5)$. The circles show the empirical data. The scales are the same as in Ref. [4].

$\sigma = 0.0508$. The solid line represents the result of simple fitting of the empirical data by the ordinary symmetrical Lévy α-stable distribution:

$$P_{\text{Levy}}(\Delta x/\sigma) = \frac{1}{\pi\sigma} \int_0^{\infty} \mathrm{d}y \cos(y\,\Delta x/\sigma) \mathrm{e}^{-\gamma y^{\alpha'}} \tag{27}$$

of index $\alpha' = 1.40$ and the scale factor $\gamma = 0.00375$ (see Fig. 2, Ref. [4]). Approximately, exponential deviation from the symmetrical α-stable ($\alpha \equiv \alpha' = 1.40$) Lévy distribution is observed for $|\Delta x|/\sigma \geqslant 6$.

For numerical calculations we rewrite Eq. (24) as follows:

$$L_{\alpha,\mu}(\eta, \Delta\tau) = D \int_0^{\infty} \mathrm{d}z \left(1 - \exp\{-S_\mu^\alpha \eta^\alpha |s_\mu(z + \Delta\tau) - s_\mu(z)|^\alpha\}\right), \tag{28}$$

where $\Delta\tau = \Delta t/\delta$ is the dimensionless time delay, the dimensionless parameters D and S_μ are defined, respectively, as $D = v\delta$, $S_\mu = b\delta^\mu/\sigma$, and the function $s_\mu(z)$ is

$$s_\mu(z) = z^\mu E_{1,1+\mu}(-z). \tag{29}$$

Fig. 1 is a plot of $P_{\alpha,\mu}(\Delta x/\sigma, \Delta\tau)$ given by

$$P_{\alpha,\mu}(\Delta x/\sigma, \Delta\tau) = \frac{1}{\pi\sigma} \int_0^{\infty} \mathrm{d}\eta \cos(\eta \cdot \Delta x/\sigma) \times \mathrm{e}^{-L_{\alpha,\mu}(\eta,\Delta\tau)}, \tag{30}$$

with $L_{\alpha,\mu}(\eta, \Delta\tau)$ defined by Eq. (28) for $D = 1.4$, $S_\mu = 0.12$ and $\Delta\tau = 0.5$. Decreasing α ($\alpha < 2$) and μ ($\mu < 1$) results in narrowing of the central part of the pdf and raising of the tails. By varying the fractality parameters α and μ it is possible to fit the empirically observed pdf.

It is easy to see that the new fractional pdf $P_{1.85,0.6}(\Delta x/\sigma, \Delta\tau = 0.5)$ when $\alpha = 1.85$ and $\mu = 0.6$ fits well the central part and the tails of the empirically observed for real time delay 1 min probability distribution of the S&P 500 index returns.

5. Conclusions

The high-frequency data for financial markets have made it possible to investigate market dynamics on timescales as short as 1 min, a value close to the minimum time needed to perform transaction in the market. The empirical data display non-Gaussian and non-Lévy long-tail distributions which cannot be explained in the framework of the traditional Gaussian- or Lévy-based approaches.

We have elaborated a fractality concept in financial mathematics and engineering. Our main assumption is that the fluctuating market phenomena can be adequately described by means of the fractional, non-Gaussian, long-range dependence stochastic process. To describe the dynamics of the price, we have introduced the new fractional stochastic differential equation, random force being the generalized "shot noise", each pulse of which has a random amplitude with the α-stable Lévy distribution. As a result we have the general expression for the fractional pdf of returns (see Eq. (20)). Theoretical predictions have been compared with the empirical data of the S&P 500 index during the 6 year period from January 1984 to December 1989. The analysis confirms that the central part and the asymptotic tails' behavior of the S&P 500 index distribution are well fitted by the developed fractional pdf.

The new fractional pdf has two fractality parameters μ and α. The parameter μ describes the dynamical memory effects in the market stochastic evolution, while the Lévy index α describes the long-range dependencies of external impacts on market dynamics. By comparing the empirical and the theoretical distributions we can conclude that fluctuating market phenomena have fractional behavior.

Acknowledgements

I thank Dr. Alexander Zhukov (NSC KIPT, Ukraine) for his valuable comments and discussions.

References

[1] L. Bachelier, Ann. Sci. École Norm. Suppl. 3 (1900) 21.
[2] F. Black, M. Scholes, J. Political Econ. 81 (1973) 637.
[3] B.B. Mandelbrot, J. Business 36 (1963) 294.
[4] R.N. Mantega, H.E. Stanley, Nature 376 (1995) 46.
[5] U.A. Muller, M.M. Dacorogna, O.V. Pictet, in: R.J. Adler, R.E. Feldman, M.S. Taqqu (Eds.), A Practical Guide to Heavy Tails, Birkhäuser, Basel, 1998, p. 283.
[6] P. Gopikrishnan, V. Plerou, L.A. Nunes Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E 60 (1999) 5305.
[7] P. Lévy, Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937.
[8] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman, San Francisco, 1983.
[9] E. Fama, J. Business 35 (1963) 420.
[10] R.N. Mantega, H.E. Stanley, Phys. Rev. Lett. 73 (1994) 2946.
[11] I. Koponen, Phys. Rev. E 52 (1995) 1197.
[12] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives and their Applications, Nauka i Tekhnika, Minsk, 1987. Translation by Harwood Academic.
[13] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[14] A. Erdelyi (Ed.), Higer Transcendental Functions, Vol. 3, McGraw-Hill, New York, 1955.
[15] B.B. Mandelbrot, J.W. van Ness, SIAM Rev. 10 (4) (1968) 422.
[16] N. Laskin, e-print, http://xxx.lanl.gov/cond-mat/9909427.