Stochastic Theory of Foreign Exchange Market Dynamics

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Abstract

A new stochastic theory of a foreign exchange markets dynamics is developed. As a result we have the new probability distribution which well describes statistical and scaling dependencies "experimentally" observed in foreign exchange markets in recent years. The developed dynamic theory is compared with well-known phenomenological Levy distribution approach which is widely applied to this problem. It is shown that the developed stochastic dynamics and phenomenological approach based on the Levy distribution give the same statistical and scaling dependencies.

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1 Introduction

Statistical behavior of the foreign exchange (FX) markets and price fluctuations in currency have been the subject of studies in recent years [1], [2]. The high-frequency data for financial markets has made it possible to investigate market dynamics on timescales as short as 1 min, a value close to the minimum time needed to perform transaction in the market. It was observed

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that the short-term price fluctuations in FX market, for example, between US dollar and German mark, has the same statistical behavior as the velocity differences in hydrodynamic turbulence [2].

Probability distributions in turbulence well fit the experimental data by superposition of the Gaussians with log-normal distributions of its variances [2],[3]. The convergence of the velocity differences distributions toward a Gaussian shape corresponds to a decrease of the log-normal variance for increasing of spatial distances Δr. In the FX fluctuation dynamics the statistical distributions of the price difference separated by time Δt was elaborated by the theoretical model of a Levy walk or Levy flight [1],[4].

I have developed a new stochastic dynamic theory for the FX market. The Langevin type stochastic differential equation is proposed. I introduce the random force with some general characteristics in order to fit the theoretical predictions of my model with observed statistical dependencies.

It is shown that the developed model describes exactly the "experimental" data of the dynamics of a price index of the New York Stock Exchange. The probability distributions of the Standard & Poor's 500 index differences Δx are reproduced exactly by the $P_{Laskin}(\Delta x, \Delta t)$. The new model describes well the scaling behavior of the "probability of return" $P_{\Delta t}(0)$ as a function of Δt (for definitions, see [1]).

2 Dynamic model of price changes

We will describe the dynamics of the FX price $x(t)$ by the stochastic differential equation

$$\dot{x}(t) = F(t), \tag{1}$$

where $F(t)$ is the random force. The quantity which were interested about is the price differences separated by the time scale Δt,

$$\Delta x = x(t + \Delta t) - x(t), \tag{2}$$

The distribution function of stochastic process Δx is defined as follow

$$P(\Delta x, \Delta t) = < \delta(\Delta x - \int_{t}^{t+\Delta t} d\tau F(\tau)) >, \tag{3}$$

where <...> means the averaging over the all possible realizations of the random force $F(t)$. Let us construct the general stochastic force $F(t)$ by the following way

$$F(t) = \sum_{k=1}^{n} a_k \varphi(t - t_k), \tag{4}$$

here a_k are the random amplitudes, $\varphi(t)$ is the response function, t_k are the homogeneously distributed (on time interval $[0, T]$) moments of time, the number n of which obeys the Poisson law.

Thus, the averaging includes three statistically independent averaging procedures:

1. Averaging over random amplitudes a_k, $< ... >_{a_k}$,

$$< ... >_{a_k} = \int da_1 ... da_n P(a_1, ..., a_n) ..., \tag{5}$$

where $P(a_1, ..., a_n)$ is the probability distribution of amplitudes a_k.

2. Averaging over t_k on time interval T,

$$< ... >_T = \frac{1}{T} \int_0^T dt_1 ... \frac{1}{T} \int_0^T dt_n .... \tag{6}$$

3. Averaging over random numbers n of time moments t_k,

$$< ... >_n = \sum_{n=0}^{n} \frac{\overline{n}^n}{n!} e^{-\overline{n}} ..., \tag{7}$$

where $\overline{n} = \nu T$ and ν is the density of points t_k on time interval T.

Taking into account the definition Eq.(3) and performing the averaging in accordance with Eqs.(5)-(7) we will have

$$P_{Laskin}(\Delta x, \Delta t) = \frac{1}{\pi} \int_0^{\infty} d\xi \cos(\xi \cdot \Delta x) e^{-L(\xi, \Delta t)}, \tag{8}$$

where we put notation

$$L(\xi, \Delta t) = \nu\delta \int_0^{\frac{\sigma\delta}{\sqrt{2}}(1 - e^{-\Delta t/\delta})} \frac{du}{u}(1 - e^{-u^2}). \tag{9}$$

For simplicity the Eq.(8) is considered for the market currency situation when $P(a_1, ..., a_n)$ is factorized as follow

$$P(a_1, ..., a_n) = \prod_{k=1}^{n} P_1(a_k), \tag{10}$$

where the Gaussian distribution $P_1(a)$ is given by

$$P_1(a) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{a^2}{2\sigma^2}). \tag{11}$$

We also choose the response function in the form

$$\varphi(t) = e^{-|t|/\delta}. \tag{12}$$

The response function describes the influence of a piece of information which has become available at the delay time t on the decision of a trader to propose or accept a price change. The parameter δ is the character scale of the time delay. We also have considered the response function in the form

$$\varphi(t) = \frac{1}{1 + (|t|/\delta)^{\beta}}, \tag{13}$$

where β is a new parameter.

For example, for $\beta = 1$ we have

$$P_{Laskin}(\Delta x, \Delta t; \beta = 1) = \frac{1}{\pi} \int_0^{\infty} d\xi \cos(\xi \cdot \Delta x) e^{-L_\beta(\xi, \Delta t; \beta=1)}, \tag{14}$$

where

$$L_\beta(\xi, \Delta t; \beta = 1) = \nu\Delta t \int_0^{\ln\frac{\Delta t+\delta}{\delta}} \frac{dz e^z}{(e^z - 1)^2} (1 - e^{-\frac{\sigma^2\delta^2\xi^2}{2} z^2}). \tag{15}$$

Thus, we derive the new $P_{Laskin}(\Delta x, \Delta t)$ distribution (see Eqs. (8), (9)) starting from the differential stochastic equation Eq.(1).

As it was mentioned the Levy stable distribution is widely applied to this problem.

The comparison between $P_{Laskin}(\Delta x, \Delta t)$ transformed to the form

$$P_{Laskin}(\Delta x, \Delta t) = \frac{1}{\pi} \int_0^\infty dy \cos(y \cdot \Delta x) \times, \quad (16)$$

$$\times \exp\left\{ -\frac{1}{2} D \int_0^{\tau^2(\Delta t) y^2} \frac{dz}{z}(1 - e^{-z}) \right\},$$

the Levy stable distribution [1], [4], [5]

$$P_{Levy}(\Delta x, \Delta t) = \frac{1}{\pi} \int_0^\infty dy \cos(y \cdot \Delta x) e^{-\gamma \Delta t y^\alpha}, \quad (17)$$

and the Gauss distribution

$$P_{Gauss}(\Delta x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-x^2/2\sigma^2\}. \quad (18)$$

is shown in Fig.1.

Figure 2 is a comparison of scaling properties of $P_{Laskin}(0, \Delta t)$, and $P_{Levy}(0, \Delta t)$.

It is interesting that the *Laskin* stochastic dynamics and phenomenological approach based on the *Levy* distribution give the same scaling for the probability of return $P_{\Delta t}(0)$ of the S&P 500 index variations as a function of the Δt [1].

3 General FX rate dynamic model

The main goal of the developed theory is to predict the behavior of currency exchange market in the future and based on these prediction to develop the efficient currency market strategy.

As we will see further it is useful to generalize the Eqs.(1), (4) in following way

$$\dot{x}\ (t) = -\lambda x + \sum_{k=1}^{n} a_k \varphi(t - t_k) \quad (19)$$

where λ is a "price dissipative coefficient" the financial mean of which will be discussed further.

Let us define the new PDF $P_{Laskin}(x,t)$ as follow

$$P_{Laskin}(x,t) =< \delta(x - x(t,x_0)) >_{a,T,n} \tag{20}$$

where $x(t,x_0)$ is the formal solution of the Eq.(19). and

$$< ... >_{a,T,n}= \int da_1...da_n P(a_1,...,a_n) \sum_{n=0}^{n} \frac{\overline{n}^n}{n!} e^{-\overline{n}} \frac{1}{T} \int_0^T dt_1 ... \frac{1}{T} \int_0^T dt_n ...$$

Using the Eqs.(19), (20) it is easy to obtain the evolution equation for PDF $P_{Laskin}(x,t)$

$$\frac{\partial P_{Laskin}(x,t)}{\partial t} = \lambda \frac{\partial}{\partial x} x P_{Laskin}(x,t) + \tag{21}$$

$$+\nu \int da P_1(a) \int_{t_0}^{t} dt' \varphi(t-t') \left\{ e^{-a \int_{t_0}^{t} d\tau e^{\lambda(t-\tau)} \varphi(\tau - t') \frac{\partial}{\partial x}} - 1 \right\} P_{Laskin}(x,t)$$

The initial condition for this equation has the form

$$P_{Laskin}(x,t) =< \delta(x - x(t,x_0)) >_{a,T,n} \tag{22}$$

The Eq.(21), (22)will serve as the main equations to predict the behavior of FX market in the future.

It is easy to see that the solution of the problem Eqs.(21) and (22) can be written as

$$P_{Laskin}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\xi(x - e^{\lambda(t-t_0)} x_0)} \times \tag{23}$$

$$\times \exp \left\{ -\nu \int_{t_0}^{t} dt' \left(1 - W(\xi \int_{t_0}^{t} d\tau e^{\lambda(t-\tau)} \varphi(\tau - t')) \right) \right\}$$

where $W(\xi)$ is characteristic function of the random amplitude a. It is well-known [5] that the PDF $P_1(a)$ and characteristic function $W(\xi)$ are connected each other

$$P_1(a) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\xi e^{i\xi a} W(\xi)$$

and

$$W(\xi) = \int\limits_{-\infty}^{\infty} da e^{-i\xi a} P_1(a)$$

The Eqs.(21) and (22) allow one to study the evolution problems in FX currency market. Some analytically solvable evolution problems will be demonstrated in the next publication.

4 Conclusions

The new dynamic stochastic theory of FX currency market dynamics is developed. We have established the new *Laskin* distribution which well describes the observed statistical dependencies of the price difference separated by time scale Δt and the *Laskin* PDF which allows elaborate evolution of the currency market and predict behavior of market dynamics.

The theoretical predictions based on the *Laskin* distribution are compared with phenomenological the *Levy* distribution based approach and observed statistical dependencies. It is interesting that the *Laskin* stochastic dynamics and phenomenological approach based on the *Levy* distribution give the same statistical and scaling dependencies.

References

[1] R.N. Mantega and H.E. Stanley, Scaling behaviour in the dynamics of economic index, Nature, v.**376** 1995, p.46-49.

[2] S. Ghashghaie, W.Breymann, J. Peinke, P.Talkner, Y.Dodge, Turbulent cascades in foreign exchange markets, Nature, v.**381**, 1996, p.767-770.

[3] B. Castaing, Y. Gagne and E. Hopfinger, Velocity probability density functions of high Reynolds number turbulence, Physica **D46**, 1990, p.177-200

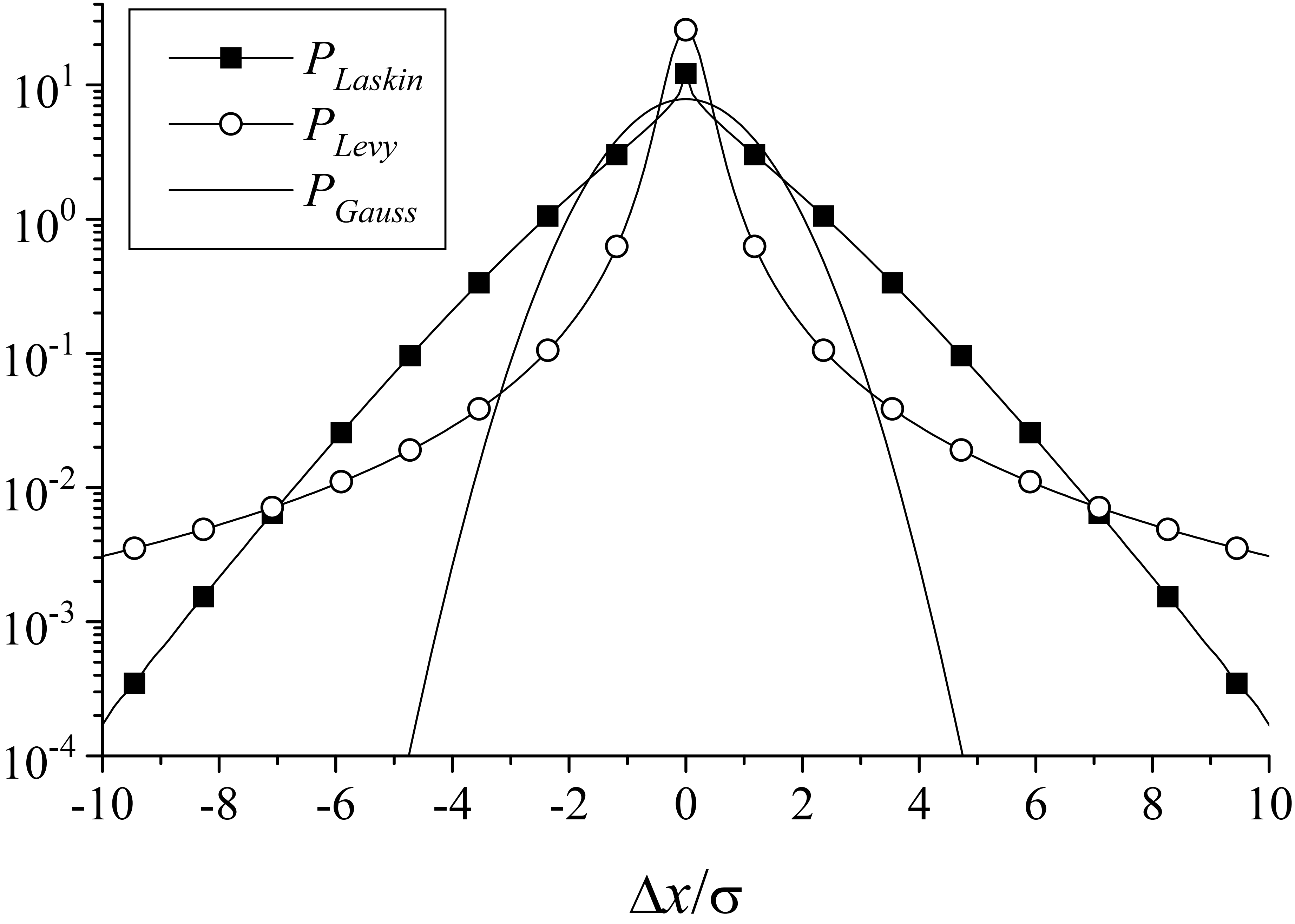
[4] P. Levy, *Theorie de l'addition des variables aleatoires*, Gauthiere-Villars, Paris, 1937

[5] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1971)

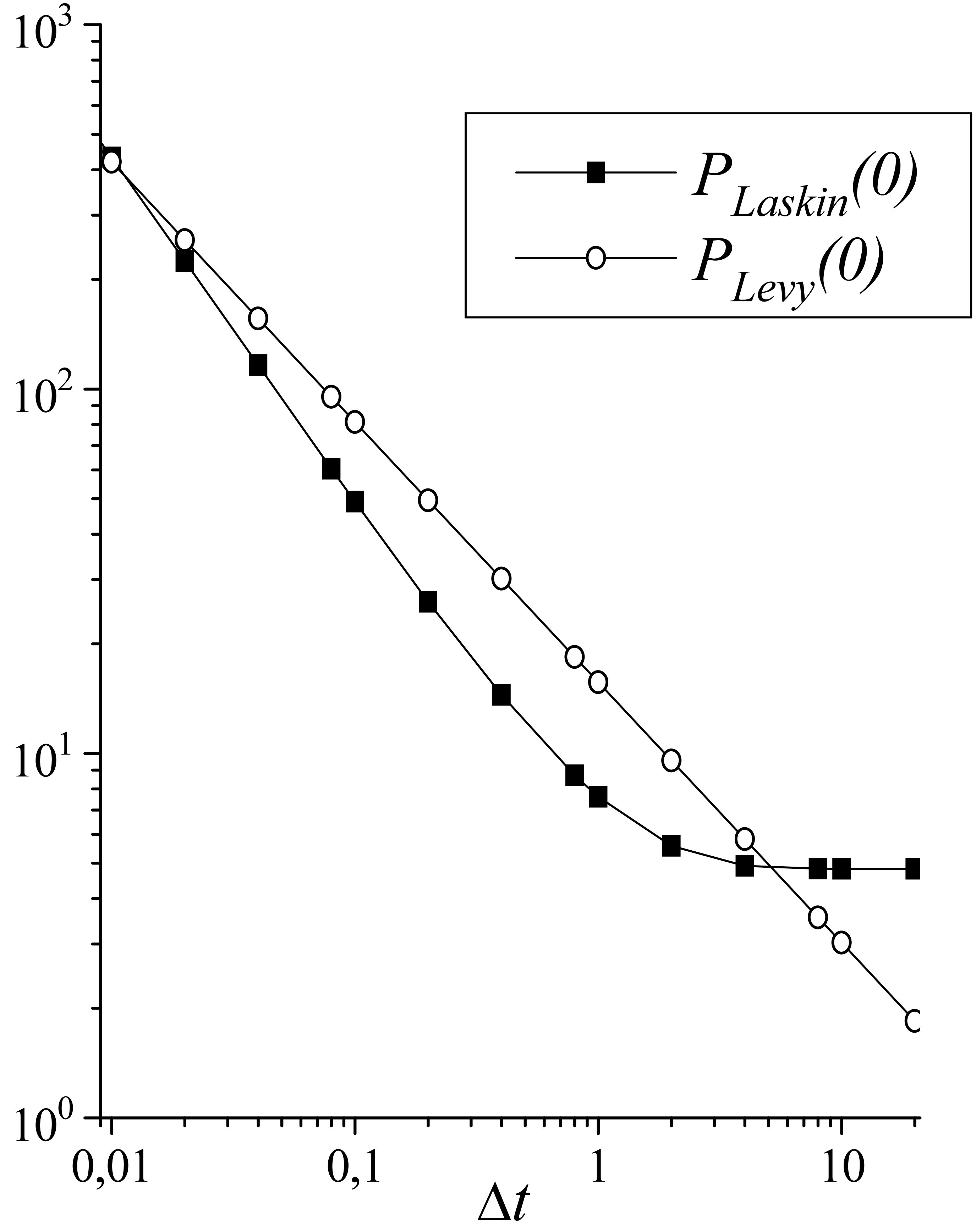
5 Figure captions

Fig.1. Comparison between $P_{Laskin}(\Delta x, \Delta t)$, $P_{Levy}(\Delta x, \Delta t)$ and $P_{Gauss}(\Delta x)$ (see the definitions Eqs.(16), (17), (18)). For $P_{Laskin}(\Delta x, \Delta t)$ the parameters are $\Delta t = 0.5$, $D = 1.4$, $S = 0.16$ and $\tau(\Delta t) = S\cdot(1-e^{-\Delta t})$. For $P_{Levy}(\Delta x, \Delta t)$ the parameters are $\alpha = 0.00375$ and $\gamma = 1.4$. For $P_{Gauss}(\Delta x)$, $\sigma = 0.0508$ (see, [1]).

Fig.2 Scaling dependencies of $P_{Laskin}(0, \Delta t)$ and $P_{Levy}(0, \Delta t)$ "probabilities of return". All parameters are the same as for Fig.1.



P_{Laskin}
P_{Levy}
P_{Gauss}
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
-10
-8
-6
-4
-2
0
2
4
6
8
10
$\Delta x/\sigma$



10^3
10^2
10^1
10^0
0,01
0,1
1
10
Δt
$P_{Laskin}(0)$
$P_{Levy}(0)$