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CONTENTS

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ON THE CONTINUATION OF THE IDEA OF HEAVISIDE IN THE OPERATIONAL CALCULUS[1)]

SUMMARY

This paper is an extension of the classical methods of operational calculus with respect to theoretical assumptions, as well as to practical applications. The foundations of this method were published by the author in 1957 [1] [2)]. The paper aims at a uniform treatment of operational methods adopted to different types of problems, such as the solving of differential and difference equations with constant coefficients, Euler equations, difference-differential equations, Bernoulli equations and other nonlinear equations. The presented method is based on the fundamental notions of functional analysis and is a continuation of the idea of Heaviside.

1. INTRODUCTION

Heaviside, a British engineer, is considered the originator of the operational calculus, in spite of there being earlier papers by Cauchy on this subject. Heaviside simply presented, in differential equations, $p = \frac{d}{dt}$ thus reducing them to algebraic equations and solving them by algebraic methods. When criticized by mathematicians for the lack of justification in this procedure, he would answer: *"Why should I resign from dinner only because I don't know the process of digesting."*

Nowadays the operational calculus is evidently a fully justified discipline. The modern interpretations of the operational calculus largely depart from the original version of Heaviside's idea. They are based on integral transformations, namely, the so-called Laplace transformations.

In order to attain possibly the greatest generality of a given mathematical method, it is important in the argumentation of the method to make use of only essentially necessary restricting assumptions. It turns out that the operational methods based on integral transformations do not satisfy this postulate. They impose, for instance, restrictions con-

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1) Reprinted by the kind permission of the Journal of the Franklin Institute (Vol. 276, No. 5, 1963, pp. 411—440).

2) The boldface numbers in parentheses refer to the references appended to this paper.

cerning the class of functions being transformed, and such restrictions, as it appears, are not indispensary in the justification of the calculus.

In electrical engineering, this method can be a basis for broader than hitherto conceived methods of analysis and synthesis of electric systems constructed of elements which constitute a physical realization of operators understood in a general way.

2. THEORETICAL FOUNDATIONS

2.1. Algebra of Operators

2.1.1. Determination of the Basic Operation

Assume that in a certain linear space **X** on the body of complex numbers **Z** a linear operation T was determined that satisfies the following conditions:

$$T(\mathbf{X}) \subset \mathbf{X}, \tag{1}$$

$$T[\alpha x_1 + \beta x_2] = \alpha T x_1 + \beta T x_2, \tag{2}$$

where α and β are complex numbers, and x_1 and x_2 are two arbitrary elements of the space **X**. Such an operation is called *endomorphism*.

Owing to the condition (1), the operation T can be performed several times in the space **X**. The n-fold operation $T^n[x]$ is determined by the recurrent formula

$$T^n[x] = T[T^{n-1}[x]]; \qquad n = 1, 2, \ldots \tag{3}$$

Assume also that the symbol T^0 will consistently denote the identity operator possessing the following property

$$T^0[x] = x. \tag{4}$$

Owing to this property the operator T^0 will also be written as the number 1:

$$T^0 = 1. \tag{5}$$

Assume that the set of the results of power operations $x_i = T^i x$, constitutes a system of linearly independent elements, and thus the following condition is also satisfied

$$\sum_{i=0}^{n} \alpha_i T^i x = 0 \equiv \alpha_0 = \alpha_1 = \ldots \alpha_n = 0 \qquad \text{or} \qquad x = 0. \tag{6}$$

where n is any natural number.

2.1.2. *Polynomial Operators*

Consider now the set **P** of polynomial operators

$$W(T) = \alpha_0 + \alpha_1 T + \ldots + \alpha_n T^n; \quad \alpha_i \in Z. \tag{7}$$

For polynomial operators determine the equality, the sum, and the product in a manner analogous for algebraic polynomials.

The polynomial operator (7) for which $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$ is called the zero operator and is written simply as the number zero.

The result of the operation $W(T)x$, determined by means of a polynomial operator, is specified by the formula

$$W(T)x = \sum_{i=0}^{n} \alpha_i T^i x. \tag{8}$$

It may be noted that the following relations hold valid for two arbitrary polynomials W_1 and W_2

$$\begin{aligned} W_1 x + W_2 x &= (W_1 + W_2) x, \\ W_1 (W_2 x) &= (W_1 W_2) x. \end{aligned} \tag{9}$$

Theorem 1. The set **P** of polynomial operators forms a commutative ring with no zero divisors.

Proof: This is almost immediate. Since on the set of polynomial operators we have determined operations analogous as for algebraic polynomials, the set **P** forms an Abelian group in respect of addition. Moreover, the multiplication of polynomial operators is commutative, and also associative and distributive with respect to addition. The set **P** forms then a commutative ring. This has no zero divisors because of the condition (6), as we have

$$Wx = 0 \equiv W = 0 \quad \text{or} \quad x = 0, \tag{10}$$

and

$$W_1 W_2 = 0 \equiv W_1 = 0 \quad \text{or} \quad W_2 = 0. \tag{11}$$

Conclusion: The results of two operations $W_1 x$ and $W_2 x$, where $x \in \mathbf{X}$, $x \neq 0$, are equal if and only if

$$W_1 = W_2.$$

In fact, if we assume that

$$W_1 x = W_2 x,$$

then owing to the first of the relations (9)

$$(W_1 - W_2) x = 0; \quad x \neq 0,$$

but, by virtue of (11) this equation is satisfied, if and only if

$$W_1 = W_2.$$

2.1.3. *Rational Operators*

Since the ring of polynomial operators $W(T)$ has no zero divisors, we can complete it in a simple manner so that it constitutes a quotient field by means of introducing rational operators in the form

$$\frac{P(T)}{Q(T)} = \frac{\alpha_0 + \alpha_1 T + \ldots + \alpha_n T^n}{\beta_0 + \beta_1 T + \ldots + \beta_m T^m}; \qquad Q \neq 0, \tag{12}$$

where α_1 and β_1 are complex numbers.

The result of the operation

$$\frac{P}{Q} x \tag{13}$$

determined by the rational operator will be called the element $y \in \mathbf{X}$ satisfying the equation

$$Px = Qy. \tag{14}$$

The operation (13) by virtue of *Theorem 1* is unique, that is, for two given polynomial operators **P** and **Q** and for a given element $x \in \mathbf{X}$ there exists one and only one element $y \in \mathbf{X}$ satisfying Eq. 14.

It is worth noting that operation (13) determined by the rational operator is not always feasible on the linear set **X** under consideration. The infeasibility of this operation takes place when the operational Eq. 14 is not solvable on the set **X**.

If, for example, **X** is a set of functions integrable according to Lebesque and determined on the real semi-axis $(0, \infty)$, then the equation

$$Tx(t) = 1; \quad t \geqslant 0, \quad x(t) \in \mathbf{X}$$

in determining the operation $Tx(t)$ by the formula

$$Tx(t) = \int_0^t x(\tau)\, d\tau$$

is not solvable on the set **X**, since it is required that the right-hand side of the equation be equal to 1 for $t \geqslant 0$, and this is impossible as the result of the operation

$$Tx(t) = \int_0^t x(\tau)\, d\tau$$

always represents a function which is equal to zero for $t = 0$. This equation would be solvable in the case where **X** is a set of distributions.

We may note that a sufficient condition (though not necessary) of the existence of the operation determined by the rational operator (12) is the condition $\beta_0 \neq 0$.

It can easily be proved that, with the assumption of the existence of given results, the following relations hold:

$$\frac{P}{Q}[\alpha x_1 + \beta x_2] = \alpha \frac{P}{Q} x_1 + \beta \frac{P}{Q} x_2, \tag{15}$$

$$\frac{P_1}{Q_1} x + \frac{P_2}{Q_2} x = \frac{P_1 Q_2 + P_2 Q_1}{Q_1 \cdot Q_2} x, \tag{16}$$

$$\frac{P_1}{Q_1}\left[\frac{P_2}{Q_2} x\right] = \frac{P_1 \cdot P_2}{Q_1 \cdot Q_2} x. \tag{17}$$

These relations signify that the rational operator $\frac{P}{Q}$ is a linear operator and that we can add and multiply rational operators in a formal way just as algebraic fractions. [3)]

The field of rational operators is then isomorphic with the field of rational functions. Owing to the existence of this isomorphism the rational operator (12) has a unique expansion into simple fractions of the form

$$\frac{1}{(T-\lambda)^k}, \tag{18}$$

which fact is of basic significance in the applications of the operational calculus.

Of special importance is the rational operator $\frac{1}{T^i}$. This operator will be also denoted by the symbol T^{-i}. Because of the above denotation we have

$$T^i T^{-i} = T^{-i} T^i = 1.$$

Generally speaking, the operators **P** and **Q**, the product of which is equal to unity (and therefore, the operators T^i, T^{-i}, too) will be called inverse operators. The operator T will also be denoted by letter p

$$p = \frac{1}{T}. \tag{19}$$

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3) As a result of the operation $\frac{P}{Q} x$ we can also understand an ordered pair $\left(\frac{P}{Q}, x\right)$ for which we assume Eqs. 15—17 as axioms. Thus Eq. 14 will always have a unique solution on the set of pairs $\left(\frac{P}{Q}, x\right)$. Such a generalization is not, however, necessary for the operational calculus.

2.1.4. *Generalized Operators*

Consider now a more general polynomial operator

$$W(T) = A_0 + A_1 T + \dots + A_n T^n, \tag{20}$$

where $A_0, A_1, \dots A_n$ are assumed endomorphisms commutative with the endomorphism T.

An operator A_j such that reduces any element y of the set $\mathbf{X}$ to the zero element, will be spoken of as the zero operator and denoted as $A_j = 0$. If then

$$A_i y = 0 \quad \text{with} \quad y \neq 0, \quad \text{then} \quad A_i = 0. \tag{21}$$

The polynomial operator (21) for which $A_1 = A_2 = \dots = A_n = 0$, will also be called a zero operator, and denoted as $W = 0$.

Assume that the results of the operation $x_i = A_i T^i x$ constitute a system of elements linearly independent in the sense that

$$\sum_{i=0}^{n} A_i T^i x = 0 \equiv A_0 = A_1 = \dots = A_n = 0 \quad \text{or} \quad x = 0. \tag{22}$$

In determining arithmetical operations on the operators (20) similarly as for algebraic polynomials, the set of these operators will evidently constitute a commutative ring having (owing to assumption (22)) no zero divisors. This ring can therefore be generalized to a quotient field in a similar manner as before, so that the rational operators $\frac{P}{Q}$ will be obtained. For operators of the form (20) the Formulas (15, 17) still hold, by reason of carrying out algebraic operations on rational operators in a formal way as on algebraic fractions. In particular, there can take place the expansion of the generalized operator $\frac{P}{Q}$ into simple fractions of the form

$$\frac{B}{(1 - CT)^k}, \tag{23}$$

where B and C are endomorphisms, and k is a natural number.

Generalized operators can, among other applications, be used in solving differential equations of the form

$$x^{(n)}(t) + A_1 x^{(n-1)}(t) + \dots + A_n x(t) = \alpha(t), \tag{24}$$

where $A_1, \dots A_n$ are continuous endomorphisms, and the functions $x(t)$, $\alpha(t)$ are continuous functions of the real variable t with values from the topological linear space $\mathbf{X}$. Such equations were considered in (6 and 7).

2.2. Analysis of Operators

2.2.1. Real Operators

Assume now that $\mathbf{X}$ is a topological linear space, such as the type of $\mathbf{L}^*$ Fréchet. The convergence of the sequence x_n in the Fréchet space is determined by the following axioms:

1. For certain sequences x_n formed from the elements of the space $\mathbf{X}$, the element $x \in \mathbf{X}$ is set into one-one relation and is called the limit of the sequence x_n:

$$x = \lim_n x_n \quad \text{or} \quad x_n \to x. \tag{25}$$

The sequences possessing a limit are called convergent.

2. Each sequence x_n possesses at most one limit.
3. If $x_n = x$ for $n = 1, 2, \ldots$, then $\lim\limits_n x_n = x$.
4. A subsequence of the sequence convergent to x is also convergent to x, that is, if $x = \lim\limits_n x_n$ and $m_1 < m_2 < \ldots$, then also $x = \lim\limits_n x_{m_n}$.
5. If each subsequence x_{m_n} of the sequence x_n contains a subsequence $x_{m_{k_n}}$ convergent to x, then $x = \lim\limits_n x_n$.

In addition to the above axioms, we accept also the following conditions of continuity:

$$\begin{aligned} x_n \to x, \quad & y_n \to y \supset x_n + y_n \to x + y, \\ x_n \to x, \quad & \alpha_n \to \alpha \supset \alpha_n x_n \to \alpha x. \end{aligned} \tag{26}$$

The notion of the series formed from the elements of a topological space is defined analogically as in the classical analysis. A limit of the sequence of partial sums is called the sum of the series. A series is called convergent, if the sequence of its partial sums has a limit on the set $\mathbf{X}$.

Let us now define the endomorphism T in the topological linear space $\mathbf{X}$, that is, an operation satisfying the conditions (1) and (2). Assume, moreover, that this endomorphism satisfies the following additional conditions:

$$\lim_n T x_n = T \lim_n x_n, \tag{27}$$

$$\sum_{i=0}^{\infty} \alpha_i T^i x = 0 \equiv \alpha_0 = \alpha_1 = \ldots = \alpha_i = \ldots = 0 \quad \text{or} \quad x = 0 \tag{28}$$

where $\alpha_i \in \mathbf{Z}$ and $x \in \mathbf{X}$, that is, the condition of continuity and of linear independence.

Consider now the sequence of rational operators

$$\frac{P_n}{Q_n}(T) = \frac{\sum_i \alpha_{in} T^i}{\sum_i \beta_{in} T^i}. \tag{29}$$

The sequence $\frac{P_n}{Q_n}(T)$ will be called convergent to the rational operator $\frac{P}{Q}(T)$.

$$\lim_n \frac{P_n}{Q_n}(T) = \frac{P}{Q}(T) \quad \text{or} \quad \frac{P_n}{Q_n}(T) \to \frac{P}{Q}(T),$$

if and only if the functional sequence $\frac{P_n}{Q_n}(z)$ of the complex variable z is almost uniformly convergent to the rational function $\frac{P}{Q}(z)$

$$\frac{P_n}{Q_n}(z) \Rightarrow \frac{P}{Q}(z).$$

It is worth noting that the convergence thus defined satisfies the axioms 1—5 of the convergence in the topological space, and also satisfies the conditions of continuity (26).

The sequence of operators $\frac{P_n}{Q_n}(T)$ will be called a basic sequence, if the function sequence $\frac{P_n}{Q_n}$ is almost uniformly convergent to the function $F(z)$ which need not be rational

$$\frac{P_n}{Q_n}(z) \Rightarrow F(z).$$

Definition: A class of basic sequences of rational operators $u_n = \frac{P_n}{Q_n}(T)$ determining uniquely the function of the complex variable $F(z) = \lim_{n\to\infty} \frac{P_n}{Q_n}(z)$, (by defining the equality of the sum and the product analogically as for the function $F(z)$) is called a real operator and is denoted by the symbol $F(T)$. We then have

$$\frac{P_n}{Q_n}(T) \sim F(T) \equiv \frac{P_n}{Q_n}(z) \Rightarrow F(z). \tag{30}$$

The set of real operators will be denoted by $\mathbf{R}_o$.

Subtraction and division of real operators are defined as inverse operations in relation to addition and multiplication, and the convergence of the sequence of operators $F_n(T)$ is understood in the sense of the divergence of the basic sequences $F_n(z)$. From the above accepted assumptions a conclusion immediately follows, namely, that the set of real operators $\mathbf{R}_0$ forms an isomorphic complete space with a certain subset of the complex variable function. The space $\mathbf{R}_0$ is a topological linear space of the L^* Fréchet type.

The result of the operation $F(T)x$:

$$y = F(T)\,x, \tag{31}$$

which is determined by means of the real operator $F(T) \sim \frac{P_n}{Q_n}(T)$ will be called an element y of the topological space $\mathbf{X}$, which satisfies the equation

$$y = \lim_n y_n = \lim_n \frac{P_n}{Q_n} x; \quad x, y, y_n \in \mathbf{X}, \tag{32}$$

with the assumption that $\frac{P_n}{Q_n} \to F \supset \frac{P_n}{Q_n} x \to Fx \in \mathbf{X}$.

Of course the operation (31) is not always feasible on the set $\mathbf{X}$. In this connection we shall determine sufficient conditions for the feasibility of this operation.

In order to formulate such conditions, consider a particular case when $\mathbf{X}$ is a locally convex space with the topology determined by means of the sequence of pseudonorms $\|x\|_k$ that is, non-negative functionals satisfying the conditions:

i_1) $\|x\|_k = 0$ when $x = 0$;
(i_2) from the condition $\|x\|_k = 0$ for $k = 1, 2, \ldots$ it follows that $x = 0$;
(i_3) $\|x + y\|_k \leqslant \|x\|_k + \|y\|_k$; $x, y \in \mathbf{X}$;
(i_4) $\|\alpha x\|_k = |\alpha| \cdot \|x\|_k$; $\alpha \in Z, x \in \mathbf{X}$.

With the metric

$$(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(1, \|x - y\|_k) \tag{33}$$

the space $\mathbf{X}$ is a topological linear space.

Such a space is particularly important in practical applications of the method discussed.

The pseudonorm of the operator T will be expressed by

$$\|T\|_k = \sup_{\|x\|_k \leqslant 1} \|Tx\|_k. \tag{34}$$

The operator T will be spoken of as essentially bounded if

$$\lim_n \sqrt[n]{\|T^n\|_k} = 0; \quad k = 1, 2, \ldots \tag{35}$$

Theorem 2. If the endomorphism T is an essentially bounded operator and there is $x_0 \in \mathbf{X}$ such that $\frac{P_n}{Q_n} x = \frac{P_n^*}{Q_n^*} x_0$ and $\lim_{n\to\infty} \frac{P_n^*}{Q_n^*}(z)$ is a holomorphic function in the neighborhood of the point $z = 0$, then the operation $F(T)x$ determined by means of the real operator $F(T) \sim \frac{P_n}{Q_n}(T)$ is feasible on the set $\mathbf{X}$.

Proof: The limit of the sequence $y_n = \frac{P_n^*}{Q_n^*} x_0$ is the sum of the power series of the operator T. We have

$$\left\| \sum_{i=0}^{\infty} \gamma_i T^i x_0 \right\|_k \leqslant \sum_{i=0}^{\infty} \|\gamma_i T^i x_0\|_k \leqslant \sum_{i=0}^{\infty} |\gamma_i| \, \|T^i\|_k \, \|x_0\| \, k < M_k \sum_{i=0}^{\infty} \frac{1}{r_1^i} \|T^i\|_k ;$$

$$k = 1, 2, \ldots; \qquad r_1 < r, \tag{36}$$

where r is the radius of convergence of the series $\sum_{i=0}^{\infty} \gamma_i z^i$. Then, if the operator T satisfies the condition (36), the sequence y_n has the limit on the set $\mathbf{X}$ and hence follows the validity of the thesis of *Theorem 2*.

It can be noted that the thesis of *Theorem 2* is also satisfied, if

$$\lim_i \sqrt[i]{\|T^i\|_k} < r_1 < r; \qquad k = 1, 2, \ldots$$

In this case the convergence of the sequence $W_n(T)$ does not imply though the convergence $y_n = W_n(T)x$, which is always true for essentially bounded operators and which is the condition of feasibility of the operation determined by means of the real operator (30). (Condition (32).)

An essentially bounded operator, such as a Heaviside operator, for which $\mathbf{X}$ is a set of the functions $x(t)$ that are continuous (or integrable) in each interval $[0, t_0]$ of the real semi-axis and

$$Tx(t) = \int_0^t x(\tau)\, d\tau . \tag{37}$$

The pseudonorm in this case is specified by the formula

$$\|x\|_k = \sup_{(\frac{1}{k}\, k)} |x(t)|; \qquad k = 1, 2, \ldots \tag{38}$$

The essential boundedness of the Heaviside operator T follows readily from the Cauchy integral

$$T^i x(t) = \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} x(\tau)\, d\tau . \tag{39}$$

The convergence implied by the sequence of pseudonorms (38) is identical with the convergence almost uniform. The linear independence in the sense of (28) for Heaviside operators results from the Titchmarsh theorem concerning convolution.

From the formula (28) it follows that a set of real operators has no zero divisors, that is

$$F(T)\,x = 0 \equiv F(T) = 0 \quad \text{or} \quad x = 0\,. \tag{40}$$

The correctness of relations analogous to (10) can also be easily proved.

In the defining real operators the approximation method in the sense of Padé is very useful.

2.2.2. Calculation of the Results of Operations by the Iterative Method

Consider the operational equation

$$y - CTy = 1\,, \tag{41}$$

where C and T are two assumed endomorphisms.

In accord with the definition of the rational operator, this equation determines the result of the following operation

$$y = \frac{1}{1 - CT}(1)\,. \tag{42}$$

Since Eq. 41 does not always have a solution in the domain of elementary functions, a convenient way of determining the result of the operation (42) is using the iterative process.

Thus, take any element $y_0 \in \mathbf{X}$ and, knowing the operation E_1T, from the equation

$$y_1 - CT_{y_0} = 1$$

calculate the element y_1. Continuing this process we get

$$\begin{aligned} y_2 &= CT_{y_1} + 1\,, \\ y_3 &= CT_{y_2} + 1\,, \\ &\cdots\cdots\cdots\cdots \\ y_n &= CT_{y_{n-1}} + 1\,. \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{43}$$

Therefore the element y_n can be expressed by the initial element y_0 as follows

$$y_n = C^n T^n y_0 + C^{n-1} T^{n-1} + \ldots + CT + 1\,.$$

If the above iterative process is convergent in the sense of the topology assumed for the space $\mathbf{X}$, then, assuming for simplicity that $y_0 = 1$, we shall obtain a power series of the operator CT convergent to y.

In this case the following expansion of the power operator (42) is then correct

$$\frac{1}{1-CT} = \sum_{i=0}^{\infty} C^i T^i. \tag{44}$$

From the expansion (44), there follows a more general relation

$$\frac{1}{(1-CT)^k} = \sum_{i=0}^{\infty} \binom{k+i-1}{i} C^i T^i, \tag{45}$$

which is obtained by raising both sides of Eq. 44 to the power k. The approximate value of the operation determined by means of the rational operator (44) can, then, be found by calculating the finite number of terms in the series, that is,

$$\frac{1}{(1-CT)^k}\, x \cong \sum_{i=0}^{n} \binom{k+i-1}{i} C^i T^i x. \tag{46}$$

Such a procedure is equivalent to the calculation of a finite number of iterations. The Formulae (45) and (46) are of course still valid for the case when the endomorphism C is replaced by a complex number α.

2.2.3. Operations in the Space of Operators

Since the set $\mathbf{R}_0$ of real operators constitutes a linear space, we can determine on this set the endomorphism T_1

$$T_1(\mathbf{R}_0) \subset \mathbf{R}_0, \tag{47}$$

$$T_1[\alpha F_1 + \beta F_2] = \alpha T_1 F_1 - \beta T_1 F_2, \tag{48}$$

$$F_1, F_2 \in \mathbf{R}_0; \quad \alpha, \beta \in Z, \tag{49}$$

satisfying the conditions established generally in the preceding chapters. If we then impose on the endomorphism T_1 conditions ensuring the existence of the results of the operation $F^{(1)}(T_1)[\mathbf{R}_0]$,we would be able to construct a new space $\mathbf{R}_1$ of the operators $F^{(1)}(T_1)$. Of course we can continue this procedure any number of times obtaining thus, spaces $\mathbf{R}_2, \mathbf{R}_3, \ldots, \mathbf{R}_n, \ldots$.

On the set $\mathbf{R}_0$ we can determine a differential operation mapping the space $\mathbf{R}_0$ onto itself and satisfying the conditions:

$$\begin{aligned} D(F_1 F_2) &= F_1 D(F_2) + F_2 D(F_1), \\ D(\alpha F_1 + \beta F_2) &= \alpha D(F_1) + \beta D(F_2). \end{aligned} \tag{50}$$

An example of this type of operation can be a difference operation determined on the set of sequences F_n of operators by means of the Formula

$$\Delta^k F_n = \sum_{\nu=0}^{k} (-1)^\nu \frac{k!}{\nu!(k-\nu)!} F_{n+k-\nu}. \tag{51}$$

Defining the endomorphism T_1 by the Formula

$$T_1 F_n = \sum_{m=0}^{n-1} F_m, \tag{52}$$

we can solve by way of operators difference equations defined in the space of operators.

Another example of a differential operation can be the differential operation of the function $F(\lambda)$ of the real variable λ with values from the space $\mathbf{R}_0$. The derivative $F'(\lambda)$ can be then determined by the isomorphism with the derivative $F'_\lambda(z, \lambda)$ of the complex variable function dependent on the real parameter λ. The properties of the derivative $F'_\lambda(z, \lambda)$ are thus transferred to the derivative $F'(\lambda)$. It is therefore possible to define a differential equation for the space of the function $F(\lambda)$ owing to the solution of partial differential equations by way of operators.

Example. The equation

$$u_{t\lambda} - \alpha u = 0, \tag{53}$$

with the boundary conditions $u(x, 0) = 0$, $u(0, t) = f(t)$ is solved by determining the endomorphism T by Eq. 37.

If we set $F(\lambda)f = u$, the partial Eq. 53 is tranformed into an ordinary equation in the space $\mathbf{R}_0^\lambda$

$$(pF'_\lambda - \alpha F)f = 0; \quad F(0) = I = 1,$$

which yields the result

$$F(\lambda) = e^{(\alpha/p)\lambda} = e^{\alpha\lambda T},$$

and

$$u = e^{\alpha\lambda T} f.$$

The result of this operation can be simply obtained by expanding $e^{\alpha\lambda T}$ into a series power of the operator T.

3. APPLICATIONS

3.1. Heaviside Operators

Assume that the set $\mathbf{X}$ is a set of functions $f(t)$ integrable in each interval $[0, t_0]$, and determine on this set the operation $Tf(t)$ as follows:

$$Tf(t) = \int_0^t f(\tau)\, d\tau . \tag{54}$$

From the Definition (54) it follows immediately that the operation $Tf(t)$ is additive and homogeneous, and the relations (2) are therefore satisfied. It may also be noted thas this operation satisfies the relation (6), and consequently the set of polynomial operators $W(T)$ forms a ring with no zero divisors.

Substituting Eq. 54, for the function $f(t)$, its derivative, we obtain

$$Tf'(t) = f(t) - f(0)$$

and

$$f'(t) = \frac{1}{T} f(t) - \frac{1}{T} f(0)$$

or introduce $\frac{1}{T} = p$

$$f'(t) = pf(t) - pf(0). \tag{55}$$

The above formula can, by simple induction, be generalized to derivatives of higher orders, obtaining then

$$f''(t) = p^2 f(t) - p^2 f(0) - pf'(0),$$

$$\boxed{f^{(\nu)}(t) = p^\nu f(t) - \sum_{n=0}^{\nu-1} p^{\nu-n} f^{(n)}(0); \qquad \nu = 1, 2, \dots ,} \tag{56}$$

it being necessary, of course, to assume the existence of the derivatives of the function $f(t)$ to the order of inclusively.

The operator $\mathbf{p} = \frac{1}{T}$ is called the differential operator and $p^{-1} = T$ — the integral Heaviside operator.

Making use of Eq. 56, we can solve linear differential equations with constant coefficients in a way analogous to the case of the method based, for example, on Laplace transformations.

In the applications it is convenient to use the following theorem well-known in the operational calculus.

Theorem. If $h(t)$ is the result of the operation $\frac{P}{Q}$ over the unit function

$$h(t) = \frac{P}{Q}(1),$$

then the result of the operation over any integrable function $f(t)$ can be expressed by means of the integral

$$\frac{P}{Q}[f(t)] = \frac{d}{dt}\int_0^t h(t-\tau)f(\tau)d\tau. \tag{57}$$

The proof of the above theorem can be found in the literature.

The integral Eq. 57 is usually called the Duhamel integral. It should be emphasized that the method presented in the present chapter is more general than the method of Laplace transformations, in the sense that it does not impose restrictions to the "speed of increase" of the function $f(t)$.

The results of certain operations can be calculated directly from the definition formula. Assuming for instance, in Eq. 54 that $f(t) = e^{\alpha t}$, we find that

$$Te^{\alpha t} = \int_0^t e^{\alpha\tau}\,d\tau = \frac{1}{\alpha}(e^{\alpha t} - 1)$$

and

$$(1-\alpha T)[e^{\alpha t}] = 1$$

and finally after substituting $p = T^{-1}$

$$\frac{p}{p-\alpha}(1) = e^{\alpha t}. \tag{58}$$

It is convenient to agree, in the case of operations performed over the unit function, to write, instead of $\frac{P(p)}{Q(p)}(1)$, simply $\frac{P(p)}{Q(p)}$. Such an agreement being made, Eq. 58 would be written as

$$\frac{p}{p-\alpha} = e^{\alpha t}. \tag{58a}$$

There are no obstacles to apply the present interpretation of the Heaviside calculus to the problems of partial differential equations. To this end it is necessary to introduce a few irrational (irregular) operators, namely,

$$e^{-\lambda p};\quad e^{-\lambda\sqrt{p}};\quad \sqrt{p};\quad \frac{1}{\sqrt{p}}. \tag{59}$$

The application of this method to the problem of partial equations would also have advantages over the method of Laplace transformations, since it would be possible, for example, to carry out proofs of uniqueness of the solutions of these equations.

3.2. Operators of Euler Equations

Assume that the operation $Tf(t)$ is determined on the set $\mathbf{X}$ by means of the following formula

$$Tf(t) = \begin{cases} \int\limits_1^t \frac{f(\tau)}{\tau} d\tau; & t \geqslant 1 \\ 0; & t \leqslant 1. \end{cases} \tag{60}$$

It can be proved that the definition determined in this way satisfies the conditions established in Chapter 2.

If in the defining formula (60), in lieu of $f(t)$, we substitute the function $tf'(t)$, we shall obtain

$$Ttf'(t) = f(t) - f(1),$$

and after substituting $T^{-1} = p$:

$$tf'(t) = pf(t) - pf(1). \tag{61}$$

Generalizing the obtained dependence to cover derivatives of higher orders we get the following Formula [4])

$$\boxed{\begin{aligned} t^{n+1} f^{(n+1)}(t) &= p(p-1)\dots(p-n)f(t) \\ &- p(p-1)\dots(p-n)f(1) - \dots \\ &- p(p-n)f^{(n-1)}(1) - pf^{(n)}(1) \end{aligned}} \tag{62}$$

where the number n is a non-negative integer.

Let us now calculate the results of certain simpler operations.

1. Assume in the defining Formula (60) that $f(t) = 1$; then

$$T(1) = p^{-1}(1) = \int\limits_1^t \frac{d\tau}{\tau} = \ln t. \tag{63}$$

Performing the operation $T(1)$ several times we get generally

$$p^{-\nu}(1) = p^{-\nu} = \frac{\ln^\nu t}{\nu!}, \tag{64}$$

where ν is a natural number and $\ln^\nu t$ denotes the simplified version of $[\ln t]^\nu$

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[4]) The above formulae can also be derived on the basis of the Mellin transformation; however, by reason of the restrictions imposed on the class of the functions $f(t)$, the obtained generalization is less extensive.

2. Assume in Formula (60) $f(t) = t^{\alpha}$; then

$$T(t^{\alpha}) = \int_{1}^{t} \tau^{\alpha-1} d\tau = \frac{\tau^{\alpha}}{\alpha}\Big|_{1}^{t} = \frac{1}{\alpha}(t^{\alpha} - 1),$$

after elementary transformations and after substitution $p = T^{-1}$

$$t^{\alpha} = \frac{p}{p-\alpha}. \tag{65}$$

By means of the operation presented in this chapter, it is possible to calculate efficiently the Euler differential equations. As is well known, the Euler equation is called an equation of the form

$$a_0 t^n f^{(n)}(t) + \ldots + a_{n-1} t f'(t) + a_n f(t) = \varphi(t),$$

where $a_0, \cdots, a_{n-1}, a_n$ are arbitrary constants and $\varphi(t)$ is a given function of the real variable t.

The example given below illustrates the manner of solving these equations by the discussed operation.

Example. Determine the function $f(t)$ satisfying the equation

$$t^2 f''(t) - f(t) = \ln t$$

and the conditions $f(1) = 0$, $f'(1) = 1$.

Solution: By virtue of Eq. 62, we get the following operational equation

$$p(p-1) f(t) - f(t) = p[1] + \ln t.$$

Consequently,

$$f(t) = \frac{p}{p(p-1)-1} + \frac{p}{p(p-1)-1}(\ln t).$$

Further, on account of Eqs. 63 and 65

$$f(t) = \frac{\alpha_1^2 - 1}{\alpha_1^2(\alpha_1 - \alpha_2)}(t^{\alpha_1} - 1) + \frac{\alpha_2^2 - 1}{\alpha_2^2(\alpha_2 - \alpha_1)}(t^{\alpha_2} - 1) - \frac{\ln t}{\alpha_1 \alpha_2},$$

where $\alpha_1, \alpha_2 = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$.

3.3. Operators of Difference Equations

3.3.1. Difference Equations with Constant Coefficients

It is possible to solve linear difference equations with constant coefficients in a simple manner by the method of the operation $Tf(t)$.

From the defining formula

$$\underset{\lambda}{T}\Delta f(t) = f(t) - f(0), \tag{66}$$

where

$$\underset{\lambda}{\Delta} f(t) = f(t + \lambda) - f(t),$$

after substituting $T^{-1} = p$ we obtain

$$\underset{\lambda}{\Delta} f(t) = pf(t) - pf(0). \tag{67}$$

Generalizing the above formula we shall have

$$\boxed{\underset{\lambda}{\Delta^n} f(t) = p^n f(t) - \sum_{\nu=0}^{n-1} p^{n-\nu} \underset{\lambda}{\Delta^\nu} f(0)}, \tag{68}$$

where it is assumed that $\underset{\lambda}{\Delta^0} f(0) = f(0)$.

It is worth noting that the above operation — contrary to all the operations discussed earlier — is not a unique operation. On the other hand, the following can be proved.

Property. If two functions $f_1(t)$ and $f_2(t)$ satisfy simultaneously the equation $T[\underset{\lambda}{\Delta} f(t)] = x(t)$:

$$T(\underset{\lambda}{\Delta} f_1) = x \quad \text{and} \quad T(\underset{\lambda}{\Delta} f_2) = x, \tag{69}$$

then the difference of the function $f_1(t)$ and $f_2(t)$ is a periodic function with the period λ.

In fact, if we assume that the functions f_1 and f_2 satisfy Eqs. 69, then, on account of the linearity of the operation $T(f)$:

$$T(\underset{\lambda}{\Delta} f_1 - \underset{\lambda}{\Delta} f_2) = 0;$$

but it follows from the defining formula (66) that the above equality implies

$$\underset{\lambda}{\Delta} f_1 - \underset{\lambda}{\Delta} f_2 = 0,$$

that is

$$\underset{\lambda}{\Delta}(f_1 - f_2) = \underset{\lambda}{\Delta} N(t) = N(t + \lambda) - N(t) = 0,$$

where

$$N(t) = f_1(t) - f_2(t).$$

The function $N(t)$ is then a periodic function with the period λ.

On account of this property the results of the operation discussed in the present chapter are "unique" with the accuracy to periodic functions with the period λ. It should be emphasized that this property of the operation $Tf(t)$ is a natural feature of all difference operations defined on the set of continuous functions. The uniqueness of the operation $Tf(t)$ could be obtained for example, by confining the domain of the operation $Tf(t)$ to a set of step functions, or by assuming that $t = 0, 1, 2, \cdots$.

Let us now calculate the results of certain simpler operations.

1. Assume that in the defining formula (66) $f(t) = (\alpha + 1)^{t/\lambda}$, then

$$\underset{\lambda}{\Delta} f(t) = (\alpha + 1)^{(t+\lambda)/\lambda} - (\alpha + 1)^{t/\lambda} = \alpha(\alpha + 1)^{t/\lambda}$$

and

$$\alpha(\alpha + 1)^{t/\lambda} = p(\alpha + 1)^{t/\lambda} - p \cdot 1$$

and finally

$$(\alpha + 1)^{t/\lambda} = \frac{p}{p - \alpha}. \tag{70}$$

2. Assume that in Eq. 66 $f(t) = t$, then

$$\underset{\lambda}{\Delta} f(t) = t + \lambda - t = \lambda$$

and

$$p^{-1}(1) = \frac{t}{\lambda}.$$

Generalizing the obtained dependence, we shall readily arrive at the following formula

$$p^{-\nu}(1) = \frac{1}{\nu!}\left(\frac{t}{\lambda}\right)^{(\nu)}, \tag{71}$$

where the expression $\left(\frac{t}{\lambda}\right)^{(\nu)}$ denotes the so called "generalized power" which is important in the difference calculus; it is defined by

$$\left(\frac{t}{\lambda}\right)^{(\nu)} = \frac{t}{\lambda}\left(\frac{t}{\lambda} - 1\right)\left(\frac{t}{\lambda} - 2\right)\ldots\left(\frac{t}{\lambda} - \nu + 1\right). \tag{72}$$

It may be noted that for the generalized power, the following formula holds

$$\underset{\lambda}{\Delta}\left(\frac{t}{\lambda}\right)^{(\nu)} = \nu\left(\frac{t}{\lambda}\right)^{(\nu-1)}, \tag{73}$$

which resembles differentiation of an "ordinary" power $\left(\frac{t}{\lambda}\right)^{\nu}$.

For the results of difference operation, it is evidently possible to set tables analogous to tables used in the method of Laplace transformations.

The application of the discussed operation to the solution of difference equations is well illustrated by the following:

Example. Find the solution of the difference equation

$$\underset{\lambda}{\Delta^2} f(t) - 3\underset{\lambda}{\Delta} f(t) + 2f(t) = 0,$$

satisfying the initial conditions

$$f(0) = 0 \quad \text{and} \quad \underset{\lambda}{\Delta} f(0) = 1.$$

Solution. In view of Eq. 68, instead of the difference equation, we may solve the following operational equation

$$p^2 f(t) - 3pf(t) + 2f(t) = p1.$$

Then

$$f(t) = \frac{p}{p-2} - \frac{p}{p-1}$$

and on account of Eq. 69:

$$f(t) = 3^{t/\lambda} - 2^{t/\lambda}.$$

3.3.2. Difference Equations with Variable Coefficients

This operation cannot be derived by means of the Laplace transformation. Consider, for example, the equation

$$x\Delta y(x) = \varphi(x).$$

This equation with the assumption of the initial condition determines uniquely the function $y(x)$. Consequently the Formula

$$T\varphi(x) = y(x) - y(1), \tag{74}$$

determines the endomorphism T. Thereby we derive a method by which it is possible to solve interesting difference equations with variable coefficients, namely equations of the type

$$a_0(x + n\lambda) \dots (x + \lambda) x\underset{\lambda}{\Delta^{n+1}} y + \dots +$$

$$+ a_{n-1}(x + \lambda) x\underset{\lambda}{\Delta^2} y + a_n x\underset{\lambda}{\Delta} y + a_{n+1} y = f(x), \tag{75}$$

where λ is the given real number and $a_0, \cdots, a_n, a_{n+1}$ are arbitrary constants.

By substituting the defining formula (74) $T^{-1} = p$ we get

$$x\underset{\lambda}{\Delta} y = py - py(\lambda). \tag{76}$$

It becomes evident that the above relation can be generalized to differences of higher orders, obtaining then

$$(x + \lambda) x\underset{\lambda}{\Delta^2} y = p(p - \lambda) y - p(p - \lambda) y(\lambda) - \lambda p\underset{\lambda}{\Delta} y(\lambda), \tag{77}$$

and generally

$$\prod_{i=0}^{n} (x + i\lambda) \Delta^{n+1} y(x) = \prod_{i=0}^{n} (p - i\lambda) y -$$

$$- \sum_{\nu=0}^{n} \prod_{i=\nu+1}^{n} p(p - i\lambda) \nu! \lambda^\nu \underset{\lambda}{\Delta^\nu} y(\lambda), \tag{78}$$

where it has been assumed that for $i > n$

$$\prod_{i=\nu+1}^{n} p(p - i\lambda) = p.$$

Now calculate the result of the operation

$$\frac{p}{p-\alpha}(1). \tag{79}$$

Note that the result of the operation (81) is the function satisfying the equation

$$x\underset{\lambda}{\Delta}y(x) - \alpha y(x) = 0 \tag{80}$$

and the initial condition $y(\lambda) = 1$.

The above equation can be solved for the "*discrete*" values of the independent variable x:

$$1\lambda, 2\lambda, 3\lambda, \ldots, k\lambda, \ldots$$

Rewriting Eq. 82 in the following equivalent form

$$y(x + \lambda) = \left(1 + \frac{\alpha}{x}\right) y(x),$$

gives the equalities

$$\begin{aligned} y(2\lambda) &= \left(1 + \frac{\alpha}{x}\right) y(\lambda), \\ y(3\lambda) &= \left(1 + \frac{\alpha}{x}\right) y(2\lambda), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y(k\lambda) &= \left(1 + \frac{\alpha}{x}\right) y(k\lambda - \lambda). \end{aligned} \tag{81}$$

As a result of multiplying the above equalities by sides, we shall have, for the initial condition $y(x) = 1$:

$$\frac{p}{p-\alpha} = \prod_{t=1}^{x-1} \left(1 + \frac{\alpha}{\lambda t}\right). \tag{82}$$

The result of the above operation is expressed especially simply in the case when $\alpha = \lambda$, then we obtain

$$\frac{p}{p-\lambda} = x. \tag{83}$$

Notice that the result of the operation

$$p^{-1}(1)$$

is the function satisfying the equation

$$\underset{\lambda}{\Delta} y(x) = \frac{1}{x}; \qquad y(\lambda) = 0.$$

It is worth noting that this equation has no solution in the set of elementary functions ([8], p. 311).

A way of practically utilizing the discussed operation is illustrated by the following:

Example. Solve the difference equation

$$(x+1)x\underset{\lambda}{\Delta^2} y(x) - 2x\underset{\lambda}{\Delta} y(x) + 2y(x) = 0; \qquad x \geqslant 1,$$

with the initial conditions $y(1) = 0$, $\Delta y(1) = 1$.

Solution. Making use of Eqs. 76 and 77, instead of the difference equation, we solve the following operational equation

$$p(p-1)y - 2py + 2y = p1.$$

Accordingly

$$y = \frac{p}{p(p-1) - 2p + 2} = \frac{p}{p-2} - \frac{p}{p-1}$$

and by virtue of Eqs. 72 and 83

$$y(x) = \prod_{t=1}^{x-1} \left(1 + \frac{2}{t}\right) - x; \qquad x \geqslant 1.$$

3.4. Operators of Difference-Differential Equations

Consider the difference-differential equation of the form

$$a_0 y^{(n)}(x+n) + \ldots + a_{n-1} y'(x+1) + a_n y(x) = 0 \tag{84}$$

where $a_0, \cdots, a_{n-1}, a_n$ are arbitrary coefficients.

By writing Eq. 84 in the form

$$y^{(n)}(x+n) = \Phi[y(x), y'(x+1), \ldots y^{(n-1)}(x+n-1)],$$

we can easily notice that if, respectively, in the intervals [0, 1], [1, 2], $\cdots$, $[n-2, n-1]$ we assume the continuous function $y(x)$ and its derivatives $y'(x), \cdots, y^{(n-1)}(x)$, thereby on the entire semi-axis $[0, \infty]$ there will be a uniquely determined continuous function $y(x)$ satisfying Eq. 84. On this account, in order to obtain uniqueness of the solution of a difference-differential equation it does not suffice to assume the values of the functions $y(x)$ and the derivatives $y^{(\nu)}(x)$ in a point, but it is necessary to impose the plot of $y(x)$ and $y^{(\nu)}(x)$ in unit intervals [0, 1], $\cdots$, $[n-2, n-1]$.

Assume that **X** is a set of the functions $y(x)$ which are continuous and integrable over the entire real axis x. On this set we define the linear operation

$$Ty'(x+1) = y(x) - y(0,1), \tag{85}$$

where $y(0, 1)$ denotes the continuous function determined by

$$y(0,1) = \begin{cases} y(x); & 0 \leqslant x < 1 \\ \text{const}; & x < 0; \quad x \geqslant 1. \end{cases}$$

Substituting in the defining Formula (85) $p = T^{-1}$, we get

$$y'(x+1) = p[y(x) - y(0,1)] = py(x) - py(0,1). \tag{86}$$

Generalizing Eq. 86 to derivatives of higher orders, we shall easily find the following relations

$$\begin{gathered} y''(x+2) = p^2 y(x) - p^2 y(0,1) - py'(1,2), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y^{(\nu)}(x+\nu) = p^\nu y(x) - \sum_{n=0}^{\nu-1} p^{\nu-n} y^{(n)}(n,n+1), \end{gathered} \tag{87}$$

where $y^{(n)}$ $(n, n+1)$ denotes the continuous function determined by

$$y^{(n)}(n,n+1) = \begin{cases} y^{(n)}(x); & n \leqslant x < n+1 \\ \text{const}; & x < n; \quad x \geqslant n+1. \end{cases} \tag{88}$$

1. Calculate the results of certain simpler operations,

$$\frac{p}{p-\alpha}(1) = P(\alpha, x). \tag{89}$$

We can easily notice that [5]

$$P(\alpha, x) = \eta(x) + \alpha(x-1)\eta(x-1) + \alpha^2 \frac{(x-2)^2}{2!}\eta(x-2) + \ldots. \tag{90}$$

For the function $P(\alpha, x)$ is the solution of the equation

$$y'(x+1) = \alpha y(x) \tag{91}$$

with the initial condition $y(x) = 1$; $x \in [0, 1]$.

2. Calculate now the results of the following operations

$$\alpha\frac{p}{p^2+\alpha^2}[1] \quad \text{and} \quad \frac{p^2}{p^2+\alpha^2}[1]. \tag{92}$$

Thus, by introducing the denotations $\sin r(\alpha, x)$ and $\cos r(\alpha, x)$ for the following continuous functions

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[5]) The function $\eta(x)$ is a step function:

$$\eta(x) = \begin{cases} 1; & x \geq 0, \\ 0; & x < 0. \end{cases}$$

$$\sin r(\alpha, x) = \frac{P(j\alpha, x) - P(-j\alpha, x)}{2j}, \tag{93}$$

$$\cos r(\alpha, x) = \frac{P(j\alpha, x) + P(-j\alpha, x)}{2},$$

then by breaking the complex operations (92) into simpler operations of the type (89), we shall readily obtain

$$\alpha \frac{p}{p^2 + \alpha^2}(1) = \sin r(\alpha, x),$$

$$\frac{p^2}{p^2 + \alpha^2}(1) = \cos r(\alpha, x). \tag{94}$$

The functions $\sin r(\alpha, x)$ and $\cos r(\alpha, x)$ are of particular importance in the problems of difference-differential equations. It can be noted that the above functions satisfy the relations

$$\sin' r(\alpha, x + 1) = \alpha \cos r(\alpha, x),$$

$$\cos' r(\alpha, x + 1) = -\alpha \sin r(\alpha, x), \tag{95}$$

resembling the differentiation of the trigonometric functions $\sin \alpha x$ and $\cos \alpha x$.

The continuity of the functions $\sin r(\alpha, x)$ and $\cos r(\alpha, x)$ results directly from Definition (93). The following relations are then valid

$$\sin r(\alpha, n + 0) = \sin r(\alpha, n - 0),$$

$$\cos r(\alpha, n + 0) = \cos r(\alpha, n - 0), \tag{96}$$

where n is a natural number.

It also follows from Eq. 93 that

$$\sin r(\alpha, 0) = 0; \quad \cos r(\alpha, 0) = 1. \tag{97}$$

A practical way of utilizing the results given in the present section is illustrated by the following:

Example. Solve the equation

$$y'''(x + 3) = 2y''(x + 2) + 9y'(x + 1) - 18y(x) = 0$$

with the initial conditions

$$y(0, 1) = y'(1, 2) = 0; \quad y''(2, 3) = 1.$$

Solution. Making use of Eq. 87 gives

$$p^3 y - 2p^2 y + 9py - 18y = p1.$$

Accordingly

$$y = \frac{p}{p^3 - 2p^2 + 9p - 18} = \frac{1}{13} \cdot \frac{p}{p - 2} - \frac{1}{13} \cdot \frac{p^2}{p^2 + 3^2} - \frac{2}{13} \cdot \frac{p}{p^2 + 3^2},$$

The relations derived above make it possible to apply the operational method in certain simpler nonlinear problems.

1. Directly from the defining Formula (98) gives

$$p^{-\nu}(1) = \frac{g^{\nu}(x)}{\nu!} . \tag{104}$$

In fact, assuming in Eq. 98 that $y(x) = 1$, we get

$$T(1) = \int_0^x g'(\xi)\,d\xi = g(x) - g(0)$$

and because of the assumption $g(0) = 0$:

$$T(1) = p^{-1}(1) = g(x). \tag{105}$$

Generalizing the obtained result we get Eq. 104.

2. Calculate the result of the operation

$$\frac{p}{p-\lambda}(1).$$

Substituting in Eq. 98 $y^{1-\alpha}(x) = e^{\lambda g(x)}$, we obtain

$$Te^{\lambda g(x)} = \int_0^x e^{\lambda g(\xi)} g'(\xi) d\xi = \frac{1}{\lambda}(e^{\lambda g(x)} - 1),$$

and hence, after elementary transformations,

$$e^{\lambda g(x)} = \frac{p}{p-\lambda} . \tag{106}$$

From the above formula it is clear that the function being the result of the operation $\frac{p}{p-\lambda}$ (1) is dependent on the weighting function $g(x)$. Assuming that $g(x) = x$, we shall get a well-known formula in the operational calculus, defining the function $e^{\lambda x}$ as a result of the operation performed over a unit function. If in another particular case we impose $g(x) = \ln x$, then the formula established earlier will be obtained

$$x^{\lambda} = \frac{p}{p-\lambda} ; \qquad x \geqslant 1$$

which was derived for the operators of the Euler equations.

Finally, if we assume that $g(x) = \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4}\right)$, which corresponds to $g'(x) = \frac{1}{\cos x}$, hence, another formula

$$tg^{\lambda}\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{p}{p-\lambda} . \tag{107}$$

and by virtue of Eqs. 89 and 94

$$y(x) = \frac{1}{13} P(2, x) - \frac{1}{13} \cos r(3, x) - \frac{2}{13} \sin r(3, x).$$

3.5. Pseudo-Nonlinear Operators

Let **X** be a set of continuous functions $y(x)$ determined on the real semi-axis $(0, \infty)$. Determine on this set the following operation

$$Tf(y) = \int_0^x f(y) g'(\xi) d\xi; \qquad y = y(x), \tag{98}$$

where $g(x)$ (weight function) is a continuous and integrable function given in advance, and satisfying the condition $g(0) = 0$.

The above operation is, of course, a linear operation, and therefore satisfies the relations (2). It can be shown that it also satisfies the other conditions established in Section 2. The operator T determined by Eq. 103 will be called the pseudo-nonlinear operator.

Assuming that the function $y(x)$ is an integrable function, then by Eq. 98 we shall get

$$\frac{1}{g'(x)} f'_u y'_x = pf(y) - pf_0, \tag{99}$$

where $f_0 = f[y(0)] = f(y)\big|_{x=0}$.

In further considerations we shall confine ourselves to the case when $f(y) = y^{1-\alpha}$, where α is a real number different from unity.

Owing to the evident equality

$$[y^{1-\alpha}(x)]' = (1-\alpha) y^{-\alpha} y' = \frac{1-\alpha}{y^\alpha} y' \tag{100}$$

we shall get

$$\boxed{\frac{1}{g'(x)} \frac{1}{y^\alpha} y' = \frac{1}{1-\alpha} py^{1-\alpha} - \frac{1}{1-\alpha} py^{1-\alpha}(0)} \tag{101}$$

where $p = T^{-1}$.

Formula (101) holds for any $\alpha \neq 1$. In a particular case, such as when $\alpha = -1$, we have

$$\frac{1}{g'(x)} yy' = \tfrac{1}{2} py^2 - \tfrac{1}{2} py^2(0). \tag{102}$$

In another particular case, namely when $\alpha = \frac{1}{2}$

$$\frac{1}{g'(x)} \frac{1}{\sqrt{y}} y' = 2p\sqrt{y} - 2p\sqrt{y}(0). \tag{103}$$

The relations derived above make it possible to apply the operational method in certain simpler nonlinear problems.

1. Directly from the defining Formula (98) gives

$$p^{-\nu}(1) = \frac{g^{\nu}(x)}{\nu\,!}\,. \tag{104}$$

In fact, assuming in Eq. 98 that $y(x) = 1$, we get

$$T(1) = \int_0^x g'(\xi)\,d\xi = g(x) - g(0)$$

and because of the assumption $g(0) = 0$:

$$T(1) = p^{-1}(1) = g(x). \tag{105}$$

Generalizing the obtained result we get Eq. 104.

2. Calculate the result of the operation

$$\frac{p}{p-\lambda}(1).$$

Substituting in Eq. 98 $y^{1-\alpha}(x) = e^{\lambda g(x)}$, we obtain

$$Te^{\lambda g(x)} = \int_0^x e^{\lambda g(\xi)} g'(\xi)\,d\xi = \frac{1}{\lambda}\left(e^{\lambda g(x)} - 1\right),$$

and hence, after elementary transformations,

$$e^{\lambda g(x)} = \frac{p}{p-\lambda}\,. \tag{106}$$

From the above formula it is clear that the function being the result of the operation $\frac{p}{p-\lambda}$ (1) is dependent on the weighting function $g(x)$. Assuming that $g(x) = x$, we shall get a well-known formula in the operational calculus, defining the function $e^{\lambda x}$ as a result of the operation performed over a unit function. If in another particular case we impose $g(x) = \ln x$, then the formula established earlier will be obtained

$$x^{\lambda} = \frac{p}{p-\lambda}\,; \qquad x \geqslant 1$$

which was derived for the operators of the Euler equations.

Finally, if we assume that $g(x) = \ln \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)$, which corresponds to $g'(x) = \frac{1}{\cos x}$, hence, another formula

$$tg^{\lambda}\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{p}{p-\lambda}\,. \tag{107}$$

It is, of course, possible to calculate in advance the results of certain operations encountered in practical problems and to set appropriate tables identical with those used in the method of Laplace transformations. Such tables would facilitate the solution of the discussed differential equations.

3. Calculate now the results of the following operations,

$$\frac{1}{p-\lambda}(1) \quad \text{and} \quad \frac{p^{-1}}{p-\lambda}(1).$$

Making use of Eq. 98 we readily find that

$$\frac{1}{p-\lambda} = T\left[e^{\lambda g(x)}\right] = \frac{1}{\lambda}\left[e^{\lambda g(x)} - 1\right], \tag{108-a}$$

and

$$\frac{p^{-1}}{p-\lambda} = \frac{1}{y^2}\left[e^{\lambda g(x)} - \lambda g(x) - 1\right]. \tag{108-b}$$

Formula (101) can, of course, be generalized to the derivative of the second order $y''(x)$. This generalization will be derived for a particular case when the weighting function $g(x)$ is a linear function: $g(x) = x$. Thus substituting

$$y^{1-\alpha} = (1-\alpha)\, y_1^{-\alpha}\, y_1'$$

in the formula

$$\frac{1}{y^\alpha}\, y' = \frac{1}{1-\alpha}\, p y^{1-\alpha} + \frac{1}{1-\alpha}\, p y^{1-\alpha}(0),$$

we get

$$y_1^{-\alpha} y_1'' - \alpha y_1^{-\alpha-1} y_1' = p y_1^{-\alpha} y_1 - p y_1^{-\alpha}(0)\, y_1'(0).$$

Now using Eq. 101 in which, according to the accepted assumption, we take into account $g'(x) = 1$, there finally results

$$y^{-\alpha} y'' - \alpha y^{-\alpha-1} y'^1 = \frac{1}{1-\alpha}\, p^2 y^{1-\alpha} - \frac{1}{1-\alpha}\, p^2 y^{1-\alpha}(0) - p y^{-\alpha}(0)\, y'(0). \tag{109}$$

It is also possible, simply, to derive a formula for the second-order derivative of $y''(x)$ by assuming that the weighting function $g(x)$ is a logarithmic function

$$g(x) = \ln x; \qquad x \geqslant 1.$$

Then, in the defining Formula (98) the lower limit of integration should be shifted from the point $x = 0$ to $x = 1$. After quite elementary transformations we get

$$x^2 y^{-\alpha} y'' - a x^2 y^{-\alpha-1} y'^2 = \frac{1}{1-\alpha}\, p(p-1) y^{1-\alpha} -$$

$$-\frac{1}{1-\alpha}p(p-1)y^{1-\alpha}(1)-py^{-\alpha}(1)y'(1). \qquad (110)$$

It can be noticed that in the case $\alpha = 0$ we obtain from Eqs. 109 and 110 the relations established earlier, namely, from Eq. 109:

$$y'' = p^2 y - p^2 y(0) - py(0)$$

and from Eq. 110:

$$x^2 y'' = p(p-1)y - p(p-1)y(1) - py'(1).$$

The examples given below illustrate the manner of practically utilizing the derived formula.

Example. Find the solution of the differential equation

$$\cos xyy' + y^2 = 1,$$

satisfying the initial condition $y(0) = 2$.

Solution. Assuming that $g(x) = \ln tg\left(\frac{x}{2} + \frac{\pi}{4}\right)$. Consequently,

$g'(x) = \frac{1}{\cos x}$, and on account of Eq. 102

$$\cos xyy' = \tfrac{1}{2}py^2 - p2.$$

We then have the following operational equation

$$\tfrac{1}{2}py^2 + y^2 = p2 + 1.$$

Hence

$$y^2 = 4\frac{p}{p+2} + \frac{2}{p+2}$$

and, on account of Eqs. 106 and 108a,

$$y^2 = 4e^{-2g(x)} - e^{-2g(x)} + 1 = 3e^{-2\ln tg\left(\frac{x}{2}+\frac{\pi}{4}\right)} + 1 = 3ctg^2\left(\frac{x}{2} + \frac{\pi}{4}\right) + 1.$$

The function being sought is, then, given by the formula

$$y(x) = \pm\sqrt{3ctg^2\left(\frac{x}{2} + \frac{\pi}{4}\right) + 1}.$$

Acknowledgment

I would like to express my gratitude to Professor L. Berg and Mr. J. Hirche of the Martin Luther Universität, Halle, for their kind remarks on this paper — in particular, regarding the correct formulation of Eqs. 90, 109 and 110.

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Translated by I. Bellert

ON FUNDATIONS OF OPERATIONAL CALCULUS [1])

The purpose of this paper is to give a uniform approach to operational methods adapted to different problems such as the solution of differential and difference equations with constant coefficients, the solution of Euler equations, difference-differential equations, and so on.

Let T be an endomorphism of the linear space X over the field C of complex numbers. The endomorphism T satisfies the condition

$$\sum_{n=0}^{N} \alpha_n T^n x \neq 0, \quad \text{for} \quad \alpha_N \neq 0 \quad \text{and} \quad x \neq 0, \quad \alpha_i \in C, \; x \in X \tag{1}$$

By this condition the ring of endomorphism

$$\sum_{n=0}^{N} \alpha_n T^n \tag{2}$$

has no divisors of zero and can be extended to the quotient field. This quotient field is isomorphic with the field of rational functions and each of its elements

$$\frac{\alpha_0 + \alpha_1 T + \ldots + \alpha_m T^m}{\beta_0 + \beta_1 T + \ldots + \beta_n T^n} \tag{3}$$

can be uniquely decomposed into partial fractions

$$\frac{1}{(T - \alpha)^k} \tag{4}$$

If we take different interpretations of the linear space X, we shall obtain either classical operational calculus or its analogies adapted to different problems. It is more convenient to denote $\frac{1}{T}$ by p.

EXAMPLES

A. Let X be a linear space of functions $x(t)$ integrable in each interval $[O, t_o]$ and

$$Tx(t) = \int_0^t x(\tau)\, d\tau \,. \tag{5}$$

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1) Bulletin de l'Académie Polonaise des Sciences (Cl. III — Vol. V, No. 9, 1957, pp. 855—858).

If the function $x(t)$ has a continuous derivative $x^{(n)}(t)$, then from we obtain

$$Tx'(t) = x(t) - x(0) \tag{6}$$

$$x^{(n)}(t) = p^n x(t) - \sum_{k=0}^{n-1} x^{(k)}(0)\, p^{n-k}, \qquad n = 1, 2, \ldots \tag{7}$$

Formula (7) permits us to solve the linear differential equations with constant coefficients, similarly as the Laplace method and other operational methods. Using the well known Duhamel operational theorem we can also solve non-homogeneous equations without any restrictions as to the speed of increase in the function.

B. Let

$$T(x)t = \begin{cases} \int_1^t \frac{x(\tau)}{\tau}\, d\tau & \text{for} \quad \tau \geqslant 1 \\ 0 & \text{for} \quad \tau < 1 \end{cases} \tag{8}$$

If the function $x(t)$ has a continuous derivative $x^{(n)}(t)$, then from

$$T[tx^1(t)] = x(t) - x(1) \tag{9}$$

we shall obtain

$$t^n x^{(n)} = p(p-1)\ldots(p-n+1)\,x(t) - p(p-1)\ldots$$
$$\ldots(p-n+1)\,x(1) - \ldots - p(p-1)\,x^{(n-2)}(1) - px^{(n-1)}(1) \tag{10}$$

Similarly as in A, formulae (10) permit us to solve the Euler equation

$$a_n t^n x^{(n)}(t) + \ldots + a_1 t x'(t) + a_0 x(t) = \varphi(t)$$

where a_n, $a_{n-1}, \ldots, a_0$ are constant coefficients and $\varphi(t)$ is a integrable function for $t \geqslant 1$.

C. Let X be a space of number sequences $x(n)$ and

$$Tx(n) = x(0) + \ldots + x(n-1).$$

From

$$T[\Delta x(n)] = x(n) - x(0), \tag{11}$$

where

$$\Delta x(n) = x(n+1) - x(n),$$

we have

$$\Delta^n x(n) = p^n x(n) - \sum_{k=0}^{n-1} \Delta^k x(0)\, p^{n-k}, \qquad n = 1, 2, 3, \ldots \tag{12}$$

Formula (12) enables us to solve difference equations with constant coefficients.

Let us consider the equation

$$n\Delta y(n) = x(n) \quad 1, 2, 3, \ldots \tag{13}$$

For a given initial condition, Equation (13) uniquely determines the sequence $y(n)$.

This means that the Formula

$$Tx(n) = y(n) - y(1), \qquad n \geqslant 1 \tag{14}$$

defines the endomorphism T.

If $p = T^{-1}$, we have

$$n\Delta x(n) = px(n) - px(1), \tag{15}$$

$$(n+1)n\Delta^2 x(n) = p(p-1)x(n) - p(p-1)x(1) - p\Delta x(1)$$

These relations enable us to solve the difference equations

$$(n+k)\ldots(n+1)n\alpha_0 \Delta^{k+1} x + \ldots + (n+1)n\alpha_{k-1}\Delta^2 x +$$

$$+ n\alpha_k \Delta x + \alpha_{k+1} x = f(n). \tag{16}$$

where $a_o, \ldots, a_{k+1}$ are arbitrary constants.

We observe that

$$\frac{p}{p-\alpha}(1) = \prod_{m=1}^{n-1}\left(1 + \frac{\alpha}{m}\right). \tag{17}$$

D. Let X be a space of functions $x(t)$ defined for $t \geqslant 0$.

We consider the difference-differential equation

$$y'(t+1) = x(t). \tag{18}$$

For the initial condition

$$y(t) = f(t) \quad \text{for} \quad t \in [0, 1],$$

Eq. (18) uniquely defines the function $y(t)$. This means that the formula

$$Tx(t) = y(t) - y(0, 1), \qquad t \geqslant 0,$$

where

$$y(0,1) = \begin{cases} f(t) & \text{for} \quad 0 \leqslant t < 1, \\ f(1) = \text{const.} & \text{for} \quad t \geqslant 1, \end{cases} \tag{19}$$

uniquely defines the endomorphism T.

If $p = T^{-1}$, then we have

$$x^{(n)}(t+n) = p^n x(t) - \sum_{k=0}^{n-1} p^{n-k} x^{(k)}(k, k+1), \tag{20}$$

where $x^{(k)}(k, k+1)$ is the following function:

$$x^{(k)}(k, k+1) = \begin{cases} x^{(k)}(t) & \text{for} \quad k \leqslant t \leqslant k+1, \\ x^{(k)}(k+1) = \text{const} & \text{for} \quad t \geqslant k+1 \end{cases}$$

Rational operations can be calculated from (19).

We put

$$\frac{p}{p-\alpha}(1) = P(\alpha, t) \tag{21}$$

It can be proved that $P(\alpha, t)$ is a continuous function defined by[2])

$$P(\alpha, t) = \eta(t) + \alpha(t-1)\eta(t-1) + \alpha^2 \frac{(t-2)^2}{2!}\eta(t-2) + \ldots \tag{22}$$

Using (20) we can solve difference- differential equations

$$\alpha_n x^{(n)}(t+n) + \ldots + \alpha_1 x'(t+1) + \alpha_0 x(t) = 0$$

by operational methods.

FINAL REMARKS

If we define in X a suitable convergence, then we shall be able to join new elements to the quotient field of endomorphisms, e.g.,

$$e^{-\lambda p}, \quad \sqrt{p}, \quad \frac{1}{\sqrt{p}}$$

Thus, we extend the domain of applications to partial differential equations

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Translated by I. Bellert

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[2]) The function $\eta(x)$ is a step function:

$$\eta(x) = \begin{cases} 1; & x \geqslant 0 \\ 0; & x < 0 \end{cases}$$

NUMERICAL OPERATOR METHOD [1])

This paper consists of three parts. In the first part a method is presented for algebraic solutions of difference equations. Elements of the algebra discussed are "numerical operators" constituting a special category of numbers. In the second part, the stability of sampled-data control systems is discussed. The third part concerns a certain numerical method which simplifies the determination of transients in electric systems excited by complex signals. The method is based on the theory presented in the first part of the paper.

INTRODUCTION

Difference equations are encountered in several problems of physics, electrical engineering and also in statistics and problems of finances. For instance, in electrical engineering, we come across the application of difference equations in the problems of electric four-poles, filters, multistage amplifiers and — above all — sampled-data control systems. Difference equations have for long been well known; they were first advanced by Broocks and Taylor (1717) [12]. In mathematics, difference equations (as also the entire difference calculus) occupy a special position on the border between algebra and analysis.

They approach analysis on account of a great similarity to differential equations, primarily, because of very similar classical methods of solving the two types of equations. On the other hand, difference equations are very close to algebra, since they are not based on the notion of limit. Accordingly, they are considered as belonging to algebra rather than to analysis.

It is likely that precisely on account of this fact several mathematicians in the middle of the last century tried to solve difference equations by algebraic methods. In this connection, from the classical methods of solving difference equations on the basis of analysis, there appeared algebraic methods sometimes called "symbolic" or "operator" methods.

Of interest here is the paper [11] by Studniczka, a Czechoslovak mathematician, (1871), in which the author, introducing certain formulae of difference calculus, treats the symbol Δ as a "number" on which arithmetical operations are performed, rather than as an operation symbol.

1) Rozprawy Elektrotechniczne (Vol. V, No 4, 1959, pp. 515—586).

It should also be noted that already in 1860 the well-known English logician, Boole [3] used the "symbolic methods" of solving difference equations. These methods were not supported by precise proofs, and therefore many mathematicians at that time and later remained cool toward this concept.

Ernest Pascal [8], one of the Italian mathematicians of the last century, expressed the following opinion on this matter: "The examples presented in this and the following sections are taken from a paper by Boole: it should, however, be emphasized that this author, like most English authors makes use of symbolic calculi usually without sufficient justification. Thus, even though these calculi yield a result in certain cases, we do not recommend them to those seeking a more elementary analytical accuracy, unless a method is formed for precise justification of passing from one formula to another. We shall try to solve all our examples absolutely excluding symbolic operations."

From the above comparison of historical data it follows that operator methods were used in difference calculus quite independently of the analogical methods in the field of differential equations, and what is more important, a few score years before Heaviside's papers.

It should be noted as a peculiarity that the correctness of the method used by Boole and Studniczka was proved on the basis of the functional transformation, not in the algebraic way; and that only after a previous justification of operator calculus by Heaviside. Consequently, the operator solution of difference equations was reduced to the determination of Laplace transforms of step functions, and the method lost its original algebraic character, its simplicity and elegance.

At present, the problems of difference equations constitute a separate branch of operator calculus, which may be called "operator calculus of step functions". In order to solve difference equations in operator calculus, we use summation or integral Laplace transformations [4], [5]. These methods lead to a quick result, but it is necessary in using them to be familiar with special branches of higher mathematics, such as for instance — apart from the theory of Laplace transformations — analytic functions. Moreover, Laplace transformations impose in advance certain restrictions on the applicability of the calculus, so that by means of these transformations, it is possible to consider only those functions which satisfy the condition of convergency of the integral or the series determining such transformations.

The author shows in the present paper that a linear difference equation may be solved algebraically, without using Laplace transformations. Instead of step functions, we shall consider "numerical operators" which are a special category of numbers. The unit of these numbers is the displacement operator q

$$Q = 0, 1, 0. 0, \dots$$

Note that the present work is an attempt to adapt, in the field of difference equations, the interpretation of operator calculus presented by Professor J. Mikusinski, a Polish mathematician.

This interpretation imparts great simplicity to the method, and facilitates its application in solving practical problems. The method calls for no knowledge of the theory of Laplace transformations or the theory of analytical functions, and therefore can be used by those who have no special mathematical background. At the same time, the method is more general since it imposes no restrictions on the class of step functions being discussed.

Apart from the theory, the present paper presents examples of practical applications of the method in electrical engineering, and particularly in the investigation of sampled-data control systems and transients. The manner of investigating sampled-data control systems by means of the method presented is taken from the paper by Tsypkin [4], in which the author makes use of what is called "summation Laplace transformation" mapping the step functions $f(n)$ on to the functions of the complex variable z, in accordance with the formula:

$$F(z) = \sum_{n=0}^{\infty} f(n) e^{-zn}.$$

The numerical operator method here gives certain simplifications and a greater generality, since it does not require a simultaneous application of two Laplace transformations (summation and integral transformations). It should, however, be emphasized that the investigation of sampled-data control systems by means of a special mathematical apparatus (such as the summation Laplace transformation or the numerical operator method) is justified only in more complicated cases. The method presented is essentially useful in the case of systems with a sampled-data feedback, whereas in dealing with an ordinary sampled-data system and periodic exciting signal, it complicates rather than simplifying. Systems with a sampled-data feedback are widely applied in control systems.

Finally, it is worth mentioning that the operations on numerical operators are isomorphic with the operations on certain subsets of Mikusiński's operators — namely, on the operators of the form $a_n e^{-sn}$. Accordingly, the method presented for solving difference equations may clearly be deduced on the basis of Mikusiński's operator calculus. However, it would then be necessary to make use of the analysis of operators, rather than the algebra, that is to say, of a notionally difficult and abstract apparatus. Thus, we come across something like a paradox: in solving notionally very simple difference equations, it would be necessary to apply a much more abstract and notionally difficult mathematical apparatus than in solving differential equations.

In defining the notion of convergence of a sequence and a series of numerical operators the author, with a view to preserving simplicity has not introduced the notions of the metric and the norm. The space metric of operators may be determined by the formula:

$$\varrho(\hat{a}, \hat{b}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|a_n - b_n|}{1 + |a_n - b_n|}.$$

By virtue of the definition of the metric, it can be shown that the space of numerical operators is a complete space — that is, each basic sequence (satisfying the Cauchy condition) of numerical operators has a limit in the set of numerical operators.

PART I

FOUNDATIONS OF THE NUMERICAL OPERATOR METHOD

THEORETICAL FOUNDATIONS

1.1. The notion of the numerical operator

1.1.1. Definition of the numerical operator

A numerical operator $\{a_n\}$ will be called an ordered system of numbers of the form:

$$\ldots 0, 0, a_{-N}, a_{-N+1}, \ldots a_{-1}, a_0, a_1, a_2, \ldots a_n, \ldots, \tag{1}$$

where N is an integral number, all the terms at the left-hand side from a_{-N} are equal to zero, and for which the equality, sum and product are defined as follows

1. Definition of the equality

Two numerical operators $\{a_n\}$ and $\{b_n\}$ are considered equal one to the other, and we write $\{a_n\} = \{b_n\}$ if and only if, when any $n : a_n = b_n$.

2. Definition of the sum

A sum of two numerical operators $\{a_n\}$ and $\{b_n\}$ is called the numerical operator $\{a_n + b_n\}$, and we write

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}. \tag{2}$$

3. Definition of the product

A product of numerical operators $\{a_n\}$ and $\{b_n)$ is called the numerical operator $\{\sum_{m=-\infty}^{\infty} a_m \cdot b_{n-m}\}$, and we write

$$\{a_n\}\{b_n\} = \left\{\sum_{m=-\infty}^{\infty} a_m \cdot b_{n-m}\right\}. \tag{3}$$

In view of the assumption that only a finite number of terms in the operator with negative indices is different from zero, the general term of the product always contains a finite number of factors.

A numerical operator will be denoted by the symbol $\{a_n\}$ or briefly $\hat{a}$. We shall sometimes use also the symbols $K\{a_n\}$ or $K\hat{a}_n$. The number K at the left-hand side of the last two symbols indicates that all the terms with the indices $n < K$ are equal to zero. We shall use the symbol $\llcorner$ to separate terms with negative indices from the remaining terms. Thus instead of writing, for example, ... 0_{-3}, 1_{-2}, 5_{-1}, 8_0, -2_1, ... we write ... 0, 1, 5, $\llcorner$ 8, − 2, ...

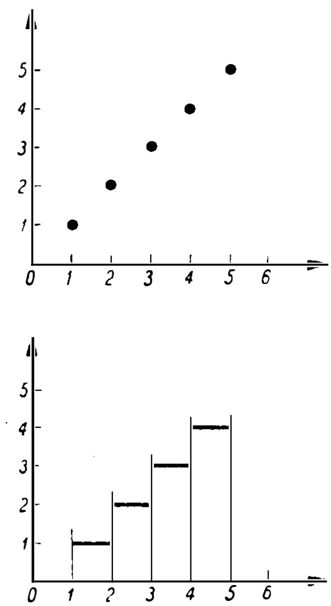


Fig. 1. Geometrical interpretation of a numerical operator

We may give a double graphical interpretation to numerical operators. Namely, we may consider that each term a_n of the numerical operator $\{a_n\}$ represents, in the rectangular coordinates, a point (n, a_n) or a rectangle having the amplitude a_n and a width equal to unity (Fig. 1).

In a particular case, if we consider operators all of whose terms corresponding to negative indices are equal to zero, the terms of the product can be represented by means of the Cauchy product, since we have

$$\{a_n\} \cdot \{b_n\} = \left\{ \sum_{m=0}^{n} a_m b_{n-m} \right\}. \tag{4}$$

From the definition of the sum and the product it follows immediately that these operations are commutative, associative and distributive. Thus we have

$$\begin{aligned} \hat{a}\hat{b} &= \hat{b}\hat{a}, \\ \hat{a}(\hat{b}\hat{c}) &= (\hat{a}\hat{b})\hat{c}, \\ \hat{a}(\hat{b}+\hat{c}) &= \hat{a}\hat{b}+\hat{a}\hat{c}. \end{aligned} \tag{5}$$

The operator $\hat{0} = \ldots 0, \ldots 0, \llcorner\, 0, 0, \ldots 0, \ldots$ is called the "zero operator".

1.2.2. Difference and quotient of numerical operators

Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary numerical operators. From the definition of the equality and the sum, it follows that there is one and only one operator $\{x_n\}$ satisfying the equation:

$$\{b_n\} + \{x_n\} = \{a_n\}.$$

This operator is called the difference of the operators $\{a_n\}$, $\{b_n\}$ and we write:

$$\{x_n\} = \{a_n\} - \{b_n\} = \{a_n - b_n\}. \tag{6}$$

The operation determining the operator $\{x_n\}$ is called the subtraction.

From the definition of the product, it follows that if the operator $\{b_n\}$ is not a zero operator, there is one and only one operator $\{x_n\}$ satisfying the equation:

$$\{x_n\} \cdot \{b_n\} = \{a_n\}, \tag{7}$$

which, by virtue of the commutative law, is equivalent to the equation:

$$\{b_n\} \cdot \{x_n\} = \{a_n\}. \tag{7a}$$

The justification of the above statement is as follows.

Assume that we have two arbitrary operators

$$\begin{aligned} {}_p\hat{a} &= \ldots 0, 0, 0, a_p, a_{p+1}, a_{p+2}, \ldots, \\ {}_k\hat{b} &= \ldots 0, 0, 0, b_k, b_{k+1}, b_{k+2}, \ldots, \end{aligned} \tag{8}$$

and

$$a_p \neq 0, \qquad b_k \neq 0.$$

In accordance with the definition of the equality and the product, equation (7) is equivalent to a system of linear equations which — after the omission of equations satisfied by identity (both sides of these equations are identically equal to zero) — takes the form

$$
\begin{aligned}
a_{2k} &= x_k b_k , \\
a_{2k+1} &= x_k b_{k+1} + x_{k+1} b_k , \\
a_{2k+2} &= x_k b_{k+2} + x_{k+1} b_{k+1} + x_{k+2} b_k , \\
& \dots\dots\dots\dots\dots\dots\dots\dots \\
a_{p-1} &= x_k b_{p-k-1} + x_{k+1} b_{p-k-2} + \dots + x_{p-k-1} b_k , \\
a_p &= x_k b_{p-k} + x_{k+1} b_{p-k-1} + \dots + x_{p-k} b_k , \\
a_{p+1} &= x_k b_{p-k+1} + x_{k+1} b_{p-k} + \dots + x_{p-k+1} b_k , \\
& \dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{9}
$$

Taking into account in the system (9)

$$a_{2k} = a_{2k+1} = \dots = a_{p-1} = 0$$

we shall obtain

$$x_k = x_{k+1} = \dots = x_{p-k-1} = 0$$

and

$$x_{p-k} = \frac{a_p}{b_k}, \quad a_{p-k+1} = \frac{a_{p+1} b_k - a_p b_{k+1}}{b_k^2}, \text{ etc.}$$

Thus we have

$$\{x_n\} = \frac{{}_p\hat{a}}{{}_k\hat{b}} = \dots 0, 0, x_{p-k}, x_{p-k+1}, x_{p-k+2}, x_{p-k+3}, \dots \tag{10}$$

and, if $p = -N < 0$, $K = -M < 0$, then

$$
\begin{aligned}
\{x_n\} = \frac{{}_{-N}\hat{a}}{{}_{-M}\hat{b}} = \dots 0, 0, x_{-(N-M)}, \\
x_{-(N-M)+1}, \dots x_{-1} \llcorner x_0, \dots x_n, \dots
\end{aligned}
\tag{10a}
$$

In the particular case in which $N = M = R$, we shall obtain

$$\{x_n\} = {}_0\hat{x} = \frac{{}_{-R}\hat{a}}{{}_{-R}\hat{b}} = \dots 0, 0, \llcorner x_0, x_1, \dots x_n, \dots$$

The numerical operator $\{x_n\}$ determined by formula (7) is called the quotient of the operator $\{a_n\}$ and by the operator $\{b_n\}$ and we write

$$\{x_n\} = \{a_n\} : \{b_n\}$$

or

$$\{x_n\} = \frac{\{a_n\}}{\{b_n\}} .$$

The operation which determines the quotient $\{a_n\} : \{b_n\}$ is called the division. A quotient exists and is uniquely determined, if and only if the diviser is different from the zero operator.

Thus, on the set of numerical operators, four operations are uniquely performable with the exception of the division by the zero operator [2]. Numerical operators thus constitute a numerical field.

Moreover, it follows from the above consideration that the index of the first non-zero term or a quotient of operators is equal to the difference of the indices of the first non-zero terms in the divident and the divisor respectively.

1.2.3. Power of numerical operators

The symbol $\hat{a}^k$ is called the "k-th power of a numerical operator" and is defined as

$$\hat{a}^k = \underbrace{\hat{a}\hat{a}\hat{a} \ldots \hat{a}}_{k}. \tag{11}$$

If $\hat{a} \neq 0$, the symbol $\hat{a}^{-k}$ is called the "k-th negative power of a numerical operator", and is defined as

$$\hat{a}^{-k} = \frac{\ldots 0, 0, \llcorner 1, 0, 0, \ldots}{\hat{a}^k}. \tag{12}$$

Moreover, we assume that

$$\hat{a}^0 = \ldots 0, 0, \llcorner 1, 0, 0, \ldots, 0, \ldots. \tag{13}$$

The definitions accepted above and the applicability of the commutative, the associative and the distributive law enable us to perform the same operations on numerical operators as on numbers.

Examples

1. $(\hat{a} + \hat{b})(\hat{a} - \hat{b}) = \hat{a}^2 - \hat{b}^2$
2. $(\hat{a} + \hat{b})^2 = \hat{a}^2 + 2\hat{a}\hat{b} + \hat{b}^2$
3. $(\hat{a} + \hat{b})(\hat{c} + \hat{d}) = \hat{a}\hat{c} + \hat{b}\hat{c} + \hat{a}\hat{d} + \hat{b}\hat{d}$
4. $\frac{\hat{a} + \hat{b}}{\hat{c} + \hat{d}} = \frac{\hat{a}}{\hat{c} + \hat{d}} + \frac{\hat{b}}{\hat{c} + \hat{d}}$; $\hat{c} + \hat{d} \neq 0$

---

[2]) Note that if we confined ourselves in advance to consider the operators determined for $n \geqslant 0$, and if we defined the quotient for those by the formula

$$\{a_n\}\{b_n\} = \left\{\sum_{m=0}^{n} a_{n-m} b_m\right\},$$

then the division would be performable in a general case, if and only if $b_0 \neq 0$.

5. $\dfrac{\hat{a}}{(\hat{b}-\hat{c})(\hat{b}-\hat{d})} = \dfrac{\hat{a}}{(\hat{c}-\hat{d})(\hat{b}-\hat{c})} + \dfrac{\hat{a}}{(\hat{d}-\hat{c})(\hat{b}-\hat{d})}$;

$\hat{b} \neq \hat{c}, \ \hat{b} \neq \hat{d}, \ \hat{c} \neq \hat{d}$.

6. If $\hat{a} \cdot \hat{b} = \hat{0}$, then $\hat{a} = \hat{0}$ or $\hat{b} - \hat{0}$.

1.2.4. Numbers versus numerical operators

Let us consider numerical operators of the form $\hat{a} = \ldots 0, 0, \llcorner a, 0, 0, \ldots$ and form the sum, the difference, the product and the quotient of such operators. In accordance with the definitions of arithmetical operations just presented, we shall obtain

$$\begin{aligned}
&(\ldots 0, 0, \llcorner a, 0, 0, \ldots) + (\ldots 0, 0, \llcorner b, 0, 0, \ldots) = (\ldots 0, 0, \llcorner a + b, 0, 0, \ldots),\\
&(\ldots 0, 0, \llcorner a, 0, 0, \ldots) - (\ldots 0, 0, \llcorner b, 0, 0, \ldots) = (\ldots 0, 0, \llcorner a - b, 0, 0, \ldots),\\
&(\ldots 0, 0, \llcorner a, 0, 0, \ldots) \cdot (\ldots 0, 0, \llcorner b, 0, 0, \ldots) = (\ldots 0, 0, \llcorner a \cdot b, 0, 0, \ldots), \qquad (14)\\
&(\ldots 0, 0, \llcorner a, 0, 0, \ldots) : (\ldots 0, 0, \llcorner b, 0, 0, \ldots) = \left(\ldots 0, 0, \llcorner \frac{a}{b}, 0, 0, \ldots\right).
\end{aligned}$$

The results of the four arithmetical operations on the numerical operators of the form $a = \ldots 0, 0, \llcorner a, 0, 0, \ldots$ correspond with the results of analogous operations on the numbers set into one-to-one correspondence with such operators. Accordingly, we say that the set of all the operators of the type $\hat{a} = \ldots 0, 0, \llcorner a, 0, 0, \ldots$ is isomorphic in respect to the four arithmetical operations with the set of numbers (real numbers, or more generally — complex numbers).

In view of this property, the operators of the type $\ldots 0, 0, \llcorner a, 0, 0, \ldots$ will be denoted by the same symbols as the numbers set into one-to-one correspondence with them, and we shall write this in the form of the equation

$$\ldots 0, 0, \llcorner a, 0, 0, \ldots = a. \qquad (15)$$

In accordance with the above agreement, the zero operator will be written as the number zero

$$\{0\} = 0, \qquad (16)$$

If, in turn, in the product $\{a_n\}\{b_n\}$ we assume $\{a_n\} = \ldots 0, 0, \llcorner a, 0, 0, \ldots$, this product may be written as formula

$$\alpha\{b_n\} = \{\alpha b_n\},$$

where a at the left-hand side of the equation is a numerical operator, while at the right-hand side it is a number.

From formula (14) it also follows that the zero operator $\{0\}$ is the modulus of addition and the operator 1 is the modulus of multiplication, as we have

$$\{a_n\} + \{0\} = \{a_n\},$$
$$\{a_n\} \cdot 1 = \{a_n\},$$

where, in accordance with (15)

$$1 = \ldots 0, 0, \ldots 0, \llcorner 1, 0, 0, \ldots 0, \ldots$$

1.3. Operations of Displacement, Summing and Difference

1.3.1. Displacement Operation

A displacement operator q will be called the following numerical operator:

$$\ldots 0, 0, \llcorner 0, 1, 0, 0, \ldots \tag{17}$$

By virtue of the definition of power, we shall obtain

$$\begin{aligned}
q^2 &= \ldots 0, 0, \llcorner 0, 0, 1, 0, 0, \ldots \\
q^k &= \ldots 0, 0, \llcorner 0, 0, 0, \ldots 1_k, 0_{k+1}, 0, \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
q^0 &= \ldots 0. 0, \llcorner 1, 0, 0, \ldots \\
q^{-1} &= \ldots 0, 1, \llcorner 0, 0, 0, \ldots \\
q^{-k} &= \ldots 0, \underset{k}{1}, \ldots \underset{k+1}{0}, \ldots \llcorner 0, 0, 0, \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{18}$$

In addition to the operator q it is convenient to introduce another displacement operator, namely

$$p = \ldots 0, 1, \llcorner 0, 0, \ldots 0, \ldots, \tag{19}$$

we then have

$$p = q^{-1}$$

and

$$p \cdot q = q \cdot p = 1. \tag{20}$$

The operators whose product is equal to 1 — for example, p and q — are called "inverse operators".

Let $\{a_n\} = \ldots 0, 0, a_{-N}, a_{-N+1}, \ldots a_{-1}, \ldots$ be an arbitrary numerical operator. The multiplication by the operator q^k is called the displacement operation.

In accordance with the definition of the product, we shall obtain

$$\begin{aligned}
q^k \{a_n\} &= \ldots 0, 0, a_{-N}, a_{-N+1}, \ldots a_{-k-1}, \llcorner a_{-k}, \ldots a_0, \ldots, \\
q^{-k} \{a_n\} &= \ldots 0, 0, a_{-N}, a_{-N+1}, \ldots a_0, \ldots a_{k-1}, \llcorner a_k, \ldots
\end{aligned} \tag{21}$$

If, in turn, we discuss the following two numerical operators

$$\{a_{n+k}\} = \ldots 0, 0, \llcorner a_k, a_{k+1}, \ldots a_{n+k}, \ldots,$$

$$\{a_{n+k+1}\} = \ldots 0, 0, \llcorner\, a_{k+1}, a_{k+2}, \ldots a_{n+k+2}, \ldots ,$$

then — taking into account (21) — we shall arrive at

$$q^{-1} \cdot \{a_{n+k}\} = \ldots 0, a_k, \llcorner\, a_{k+1}, a_{k+2}, \ldots ,$$

$$q^{-1} \cdot a_k = \ldots 0, a_k, \llcorner\, 0, 0, \ldots .$$

Hence, subtracting and substituting $q^{-1} = p$,

$$p\,\{{}^{a}_{n+k}\} - pa_k = \{a_{n+k+1}\} ,$$

or

$$\{a_{n+k+1}\} = p\,\{a_{n+k}\} - pa_k . \tag{22}$$

In a particular case in which $k = 0$

$$\{a_{n+1}\} = p \cdot \{a_n\} - pa_0 . \tag{23}$$

If in Formula (22) we assume that $k = 1$, and we take into consideration Formula (23), then we shall obtain

$$\{a_{n+2}\} = p^2\,\{a_n\} - p^2\,a_0 - pa_1 .$$

Effecting the above transformations several times, we shall arrive at

$$\{a_{n+k}\} = p^k\,\{a_n\} - p^k\,a_0 - p^{k-1}\,a_1 - \ldots - pa_{k-1} , \tag{24}$$

that is,

$$\boxed{\{a_{n+k}\} = p^k \left(\{a_n\} - \sum_{n=0}^{k-1} p^{-n}\,a_n\right),} \tag{24a}$$

where $k = 1, 2, 3 \ldots.$

In the particular case, in which

$$a_0 = a_1 = a_2 = \ldots = a_{k-1} = 0 ,$$

then

$$\{a_{n+k}\} = p^k\,a_n . \tag{25}$$

Similarly, if we are given the following two operators

$$\begin{aligned} \{a_{n-k}\} &= \ldots 0, 0, \llcorner\, a_{-k}, a_{-k+1}, \ldots a_{n-k}, \ldots , \\ \{a_{n-k-1}\} &= \ldots 0, 0, \llcorner\, a_{-k-1}, a_{-k}, \ldots a_{n-k}, \ldots , \end{aligned} \tag{26}$$

then

$$\{a_{n-k-1}\} = q\,\{a_{n-k}\} + a_{-k-1} , \tag{27}$$

whence for the case in which $k = 0$

$$\{a_{n-1}\} = q\,\{a_n\} + a_{-1} . \tag{28}$$

However, if in Formula (27) we assume $k = 1$, and we take into consideration (28), then

$$\{a_{n-2}\} = q^2\,\{a_n\} + qa_{-1} + a_{-2} .$$

Performing the above transformations several times, we shall arrive at

$$\{a_{n-k}\} = q^k \{a_n\} + q^{k-1} a_{-1} + q^{k-2} a_{-2} + \dots + a_{-k}$$

or

$$\{a_{n-k}\} = p^{-k} \{a_n\} + p^{-k+1} a_{-1} + p^{-k+2} a_{-2} + \dots + a_{-k}, \tag{29}$$

that is

$$\boxed{\{a_{n-k}\} = p^{-k} \left(\{a_n\} + \sum_{n=1}^{k} p^n a_{-n}\right).} \tag{29a}$$

In the particular case in which

$$a_{-1} = a_{-2} = \dots = a_{-k} = 0,$$

then

$$\{a_{n-k}\} = p^{-k} \{a_n\} = q^k \{a_n\}, \tag{30}$$

1.3.2. Summing operation

In what follows, we shall consider, primarily, operators the terms of which having negative indices, are equal to zero, that is to say, operators of the type

$$\{x_n\} = {}_0\{x_n\} = \dots 0, 0, \llcorner x_0, x_1, \dots x_n, \dots$$

For simplicity, we shall write these operators as follows

$$\{x_n\} = x_0, x_1, \dots x_n, \dots$$

Assume now that we are given the following numerical operator

$$\{1\} = 1, 1, 1, \dots 1, \dots \tag{31}$$

This operator will be denoted by the letter σ, and will be called the "summing operator".

By virtue of the definition of the product we have

$$\sigma^2 = \{1\} \cdot \{1\} = \left\{\sum_{m=0}^{n} 1\right\} = \{n+1\} = 1, 2, 3, \dots,$$

$$\sigma^3 = \sigma^2 \cdot \{1\} = \left\{\sum_{m=0}^{n} (n+1)\right\} = \left\{\frac{(n+1)(n+2)}{2}\right\} = 1, 3, 6, \dots$$

and more generally

$$\sigma^{k+1} = \{1\}^{k+1} = \left\{\binom{n+k}{k}\right\}, \tag{32}$$

where k is an integral number, and

$$\binom{n+k}{k} = \frac{(n+1)(n+2)\dots(n+k)}{k!}.$$

Supposse we are given an arbitrary numerical operator of the type $\{a_n\} = a_0,\ a_1,\ a_2 \ldots a_n, \ldots$. The multiplication by a summing operator will be called a "summing operation".

In accordance with the definition of the product, we shall obtain

$$\{b_n\} = \{1\}\{a_n\} = \left\{\sum_{m=0}^{n} a_m 1_{n-m}\right\} = a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots \tag{33}$$

Thus, as a result of multiplying the numerical operator $\{a_n\}$ by a summing operator, we obtain a new operator $\{b_n\}$ the terms of which are partial sums of the terms of the operator $\{a_n\}$, since we have

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= a_0 + a_1, \\ &\cdots\cdots\cdots \\ b_n &= a_0 + a_1 + a_2 + \ldots + a_n, \\ &\cdots\cdots\cdots \end{aligned} \tag{34}$$

The summing operation may be performed a number of times, for example, twice

$$\bar{\sigma}^2 \{a_n\} = \left\{\sum_{m=0}^{n}\left(\sum_{m_1=0}^{m} a_{m1}\right)\right\}$$

or three times

$$\sigma^3 \{a_n\} = \left\{\sum_{m=0}^{n}\left[\sum_{m_1=0}^{m}\left(\sum_{m_2=0}^{m_1} a_{m2}\right)\right]\right\}.$$

In performing the summing operation several times, we make use of the formula

$$\sigma^{k+1} \cdot \{a_n\} = \left\{\sum_{m=0}^{n}\binom{n-m+k}{k} a_m\right\}^{*)} \tag{35}$$

The correctness of this formula follows from the definition of the product, and from Formula (32), since we have

$$\sigma^{k+1}\{a_n\} = \left\{\binom{n+k}{k}\right\}\{a_n\} = \left\{\sum_{m=0}^{n}\binom{n-m+k}{k} a_m\right\}.$$

---

*) Note that Formula (35) is an analogue of Cauchy's integral

$$\int_0^t \frac{(t-\tau)^k}{k!} f(\tau)\, d\tau,$$

replacing the k-th integration.

1.3.3. *Difference operation*

The difference operator δ will be called the following numerical operator: 1, -1, 0,0, . . . 0, . . .

By virtue of the definition of the product, we have

$$(1, -1, 0, 0, \dots 0, \dots)(1, 1, \dots 1, \dots) = 1$$

whence,

$$\delta = \frac{1}{\{1\}} = \frac{1}{\sigma}$$

and

$$\delta \cdot \bar{\sigma} = \sigma \cdot \delta = 1 .$$

In view of the above the operators δ and σ form a pair of inverse operators.

Assume an arbitrary numerical operator $\{a_n\} = a_0,\ a_1,\ a_2, \dots a_n \dots$ The multiplication by the difference operator will be referred to as the "difference operation".

In accordance with the definition of the product, we have

$$\{b_n\} = \delta\{a_n\} = a_0, a_1 - a_0, a_2 - a_1, \dots a_n - a_{n-1}, \dots,$$

or

$$\delta\{a_n\} = \{a_n - a_{n-1}\} = \{\Delta a_{n-1}\} .$$

The difference operation may be performed several times. A difference of the second order is denoted by the symbol $\{\Delta^2 a_n\}$, and

$$\{\Delta^2 a_n\} = \{\Delta a_{n+1}\} - \{\Delta a_n\} ,$$

or, if we take into consideration the denotations $\{\Delta a_{n+1}\}$ and $\{\Delta a_n\}$,

$$\{\Delta^2 a_n\} = \{a_{n+2}\} - 2\{a_{n+1}\} + \{a_n\} ;$$

Similarly

$$\{\Delta^3 a_n\} = \{\Delta^2 a_{n+1}\} - \{\Delta^2 a_n\} ,$$

hence

$$\{\Delta^3 a_n\} = \{a_{n+3}\} - 3\{a_{n+2}\} + 3\{a_{n+1}\} - \{a_n\}$$

and, more generally

$$\{\Delta^k a_n\} = \{\Delta^{k-1} a_{n+1}\} - \{\Delta^{k-1} a_n\} ,$$

$$\{\Delta^k a_n\} = \sum_{\nu=0}^{k} (-1)^\nu \frac{k!}{\nu!\,(k-\nu)!} \{a_{n+k-\nu}\} .$$

The difference operation may conveniently be expressed by means of the displacement operator. In the case of a difference of the first order, we have

$$\{\Delta a_n\} = (p - 1)\{a_n\} - pa_0 ,$$

then, in the case of a difference of the second order

$$\{\Delta^2 a_n\} = p\,[(p-1)\{a_n\} - pa_0 - p\Delta a_0] - (p-1)\{a_n\} + pa_0 =$$
$$= (p-1)^2\{a_n\} + - p\,(p-1)\,a_0 - p\Delta a_0$$

and, more generally

$$\{\Delta^k a_n\} = (p-1)^k \{a_n\} - p \sum_{\nu=0}^{k-1} (p-1)^{k-1-\nu} \Delta^\nu a_0 ,$$

where k is an integer and $\Delta^0 a_0 = a_0$.

Examples

1. If $\{a_n\} = \{cn\}$

then

$$\{\Delta a_n\} = \{a_{n+1}\} - \{a_n\} = c(\{n+1\} - \{n\}) = c\,\{n+1-n\} = c\,\{1\} = \{c\}$$

and

$$\{\Delta^2 a_n\} = \{\Delta^3 a_n\} = \ldots = 0 .$$

2. If $\{a_n\} = \{e^{an}\}$,

then

$$\{\Delta a_n\} = \{e^{a(n+1)} - e^{an}\} = \{e^{an}(e^\alpha - 1)\} = (e^a - 1)\,\{e^{an}\},$$
$$\{\Delta^2 a_n\} = (e^a - 1)\,\{e^{a(n+1)} - e^{an}\} = (e - 1)^2\,\{e^{an}\},$$

and, more generally

$$\{\Delta^k a_n\} = (e^a - 1)^k\,\{e^{an}\} .$$

1.4. Notions of a sequence and a series of numerical operators

If every non-negative integral number k is set into one-to-one correspondence with a numerical operator $\hat{a}_k = \{a_n\}_k$, then we have a determined infinite sequence of numerical operators

$$\{\hat{a}_k\} = \hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots \hat{a}_k, \ldots \tag{36}$$

The operator $\hat{b} = \{b_n\}$ is said to be the limit of the sequence of numerical operators $\{\hat{a}_k\}$, and we write

$$\hat{b} = \lim_{k\to\infty} \hat{a}_k, \tag{37}$$

if for each n the equality

$$\lim_{k\to\infty} (a_n^k - b_n) = 0 , \tag{38}$$

holds, where a_n^k = the n-th term of the k-th operator.

The sequence of numerical operators which has a limit is called a convergent sequence. The sequence of numerical operators which has no limit is said to be divergent.

A series of numerical operators $\hat{a} = \{a_n\}$ is called the symbol

$$\hat{a}_0 + \hat{a}_1 + \hat{a}_2 + \ldots + \hat{a}_k + \ldots = \sum_{k=0}^{\infty} \hat{a}_k \cdot \tag{39}$$

The operators $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2, \ldots \hat{a}_k, \ldots$ are called constituents of the series and the operators

$$\begin{aligned} S_0 &= \hat{a}_0\,, \\ S_1 &= \hat{a}_0 + \hat{a}_1\,, \\ S_2 &= \hat{a}_0 + \hat{a}_1 + \hat{a}_2\,, \\ &\ldots\ldots\ldots\ldots \\ S_k &= \hat{a}_0 + \hat{a}_1 + \hat{a}_2 + \ldots \hat{a}_k\,, \end{aligned} \tag{40}$$

are partial sums of the series.

An infinite series is called convergent, if the sequence of partial sums $\hat{S}_k$ is convergent. The limit of a sequence of partial sums is called the sum of the series of numerical operators.

If

$$\lim_{k\to\infty} \hat{S}_k = \hat{S}\,, \tag{41}$$

then we write

$$\hat{S} = \hat{a}_0 + \hat{a}_1 + \hat{a}_2 + \ldots + \hat{a}_k + \ldots, \tag{42}$$

or

$$\hat{S} = \sum_{k=0}^{\infty} \hat{a}_k\,.$$

Let $\{a_n\}$ be an arbitrary numerical operator. If we make up a sequence of the products $a_n\, q^n$:

$$\{a_n\, q^n\} = a_0\, q^0, a_1\, q, a_2\, q^2, \ldots a_n\, q^n\,, \tag{43}$$

then we shall obtain

$$\begin{aligned} a_0\, q^0 &= a_0, 0, 0, 0, \ldots 0, \ldots, \\ a_1\, q &= 0, a_1, 0, 0, \ldots 0, \ldots, \\ a_2\, q^2 &= 0, 0, a_2, 0, \ldots 0, \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n\, q^n &= 0, 0, 0, 0, \ldots 0, a_n, 0, \ldots. \end{aligned} \tag{44}$$

In accordance with (44), the partial sums of the sequence (43) will take the form

$$\begin{aligned} \hat{S}_0 &= a_0, 0, 0, 0, \ldots 0, \ldots, \\ \hat{S}_1 &= a_0, a_1\, 0, 0, \ldots 0, \ldots. \\ \hat{S}_2 &= a_0, a_1, a_2, 0, \ldots 0, \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \hat{S}_n &= a_0, a_1, a_2, a_3, \ldots a_n, 0, 0, \ldots \end{aligned} \tag{45}$$

Be virtue of the definition of the limit of numerical operators, the sequence of partial sums $\hat{S}_n$ will have a limit which will be the operator $\{a_n\}$

$$\lim_{n\to\infty} \hat{S}_n = a_0, a_1, a_2, a_3, \dots a_n, a_{n+1}, \dots = \{a_n\}. \tag{46}$$

After taking into consideration (44) and (45), we arrive at

$$\hat{a} = \sum_{n=0}^{\infty} a_n q^n = a_0 + \sum_{n=1}^{\infty} a_n q^n. \tag{47}$$

For an operator having the general form ${}_{-N}\hat{a}_n = \dots a_{-N}, a_{-N+1}, \dots a_{-1} \llcorner a_0, \dots \dots a_n, \dots$, we shall obtain

$$\boxed{{}_{-N}\hat{a}_n = \sum_{n=-\infty}^{\infty} a_n q^n = \sum_{n=1}^{\infty} a_n p^n + \sum_{n=0}^{\infty} a_n q^n.} \tag{48}$$

Thus, every numerical operator can be represented in the form of the power series of a displacement operator.
Let us now discuss the following power series

$$\hat{S} = \sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \dots + q^n + \dots \tag{49}$$

The sequence of partial sums of this series will take the form

$$\{\hat{S}_k\} = 1, 1 + q, 1 + q + q^2, \dots 1 + q + q^2 + \dots + q^k, \dots. \tag{50}$$

We then have

$$\begin{aligned} \hat{S}_0 &= 1 \\ \hat{S}_1 &= 1 + q = 1, 1, 0, 0, 0, \dots, \\ \hat{S}_2 &= 1 + q + q^2 = 1, 1, 1, 0, 0, \dots, \\ \hat{S}_3 &= 1 + q + q^2 + q^3\, 1, 1, 1, 1, 0, 0, \dots, \\ \hat{S}_k &= 1 + q + \dots + q^k = 1, 1, 1, 1, \dots 1, 0, 0, \dots. \end{aligned} \tag{51}$$

We shall now prove that the sum of the series (49) is the expression

$$\hat{S} = \frac{1}{1-q} = \frac{1}{1, -1, 0, 0, 0, \dots}.$$

In fact, since

$$(1, -1, 0, 0, 0, \dots)(1, 1, 1, \dots 1, \dots) = 1,$$

then by virtue of the definition concerning the division of numerical operators, we shall obtain

$$\hat{S} = \frac{1}{1, -1, 0, 0, 0, \dots} = 1, 1, 1, \dots 1, \dots = \{1\}. \tag{53}$$

Hence, in accordance with the definition concerning the convergence of numerical operators (38)

$$\lim_{k\to\infty} \hat{S}_k = \hat{S}$$

and

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \ldots + q^n + \ldots = \frac{1}{1-q}. \tag{54}$$

1.5. Notion of the function of a numerical operator

If each numerical operator $\hat{z} = \{Z_n\}$ belonging to a certain set is set into one-to-one correspondence with another numerical operator $\hat{F} = \{F_n\}$, then we say that $\hat{F}(\hat{z})$ is a function on the set of numerical operators.

The exponential function $e^{\hat{z}}$ is defined by the equality

$$e^{\hat{z}} = \sum_{n=0}^{\infty} \frac{\hat{z}^n}{n!}. \tag{55}$$

The series (55) is always convergent, and therefore the exponential function is determined for any operator $\hat{z}$.

On the basis of the Definition (55), we can easily account for the following relations

$$\begin{aligned} &1.\ e^{\hat{z}_1} \cdot e^{\hat{z}_2} = e^{\hat{z}_1 + \hat{z}_2}, \\ &2.\ (e^{\hat{z}})^m = e^{m\hat{z}}. \end{aligned} \tag{56}$$

In the particular case in which $\hat{z} = q$, we shall obtain

$$e^q = 1 + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \ldots = \left\{\frac{1}{n!}\right\}. \tag{57}$$

Similarly, the trigonometric functions $\sin \hat{z}$, $\cos \hat{z}$ are defined in the field of numerical operators as the sums of the following series

$$\begin{aligned} \sin \hat{z} &= \sum_{n=0}^{\infty} \frac{(-1)^n \hat{z}^{2n+1}}{(2n+1)!}, \\ \cos \hat{z} &= \sum_{n=0}^{\infty} \frac{(-1)^n \hat{z}^{2n}}{(2n)!}. \end{aligned} \tag{58}$$

These series are always convergent, and thus $\sin \hat{z}$ and $\cos \hat{z}$ are determined for any operator $\hat{z}$.

On the basis of the Definition (58), we state that the function sin $\hat{z}$ is odd and the function cos $\hat{z}$ is even

$$\begin{aligned} \sin\hat{z} &= -\sin(-\hat{z}), \\ \cos\hat{z} &= \cos(-\hat{z}). \end{aligned} \tag{59}$$

In the field of numerical operators the equation of Euler also holds

$$e \pm iz = \cos\hat{z} \pm i\sin\hat{z}. \tag{60}$$

In the particular case, if $\hat{z} = q$, we have

$$\begin{aligned} \sin q &= \dots 0, 0, 0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, \dots, \\ \cos q &= \dots 0, 0, 1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, 0, -\frac{1}{6} \dots \end{aligned} \tag{61}$$

The trigonometric functions tan $\hat{z}$ and cot $\hat{z}$ are determined by the formulae

$$\tan\hat{z} = \frac{\sin\hat{z}}{\cos\hat{z}}, \qquad \cot\hat{z} = \frac{\cos\hat{z}}{\sin\hat{z}} \tag{62}$$

The hyperbolic functions sinh $\hat{z}$ and cosh $\hat{z}$ are defined in the field of numerical operators by the formulae

$$\sinh\hat{z} = \frac{e^{\hat{z}} - e^{-\hat{z}}}{2}, \qquad \cosh\hat{z} = \frac{e^{\hat{z}} + e^{-\hat{z}}}{2}. \tag{63}$$

By virtue of this definition, we obtain the relations

$$\begin{aligned} \sinh\hat{z} &= -i\sin i\hat{z}, \\ \cosh\hat{z} &= -\cos i\hat{z} \end{aligned}$$

and in a particular case, if $\hat{z} = q$,

$$\begin{aligned} \sinh q &= \dots 0, 0, \llcorner\, 0, 1, 0, \frac{1}{3!}, 0, \frac{1}{5!}, 0, \dots, \\ \cosh q &= \dots 0, 0, \llcorner\, 1, 0, \frac{1}{2!}, 0, \frac{1}{4!}, 0, \frac{1}{6!}, \dots. \end{aligned}$$

1.6. Examples of rational operators

Below will be given a few more important rational functions of the displacement operator. We shall use these functions in solving practical problems.

1. $\{a_n\} = \{1\} = 1, 1, 1, \dots 1, \dots.$

Taking into consideration Formula (24), we shall obtain

$$\{1\} = p(\{1\} = 1).$$

Whence

$$\{1\} = \frac{p}{p-1} = \frac{1}{1-q}.$$

$$2.\ \{a_n\} = \{n\} = 0, 1, 2, 3, 4, 5, \ldots. \tag{64}$$

In accordance with (24), we write

$$\{n+1\} = p\,(n-0).$$

Whence

$$\{n\} = \frac{\{1\}}{p-1}$$

and

$$\{n\} = \frac{p}{(p-1)^2}. \tag{65}$$

$$3.\ \{a_n\} = \{n^2\} = 0, 1, 4, 9, 16, \ldots.$$

In accordance with (24), we write

$$\{(n+1)^2\} = p\,\{n^2\}.$$

Whence

$$\{n^2\} + 2\,\{n\} + \{1\} = p\,\{n^2\},$$

or

$$\{n^2\} = \frac{\{1\} + 2\,\{n\}}{p-1},$$

and after taking into account (64) and (65), we have

$$\{n^2\} = \frac{p\,(p+1)}{(p-1)^3}; \tag{66}$$

$$4.\ \{a_n\} = \{n^3\}.$$

In accordance with (24), we write

$$\{(n+1)^3\} = p\,\{n^3\},$$

whence

$$\{n^3\} = \frac{3\,\{n^2\} + 3\,\{n\} + 1}{p-1}$$

and after taking into consideration (65) and (66) we have

$$\{n^3\} = 3\,\frac{p\,(p+1)}{(p-1)^4} + 3\,\frac{p}{(p-1)^3} + \frac{p}{(p-1)^2},$$

or

$$\{n^3\} = \frac{p}{(p-1)^4}\,(p^2 + 4p + 1). \tag{67}$$

$$5.\ \{a_n\} = \left\{\frac{n\,(n-1)}{2!}\right\}.$$

Taking into consideration Formulae (65) and (66) we shall obtain

$$\left\{\frac{n(n-1)}{2!}\right\} = \left\{\frac{n^2}{2}\right\} - \left\{\frac{n}{2}\right\} = \frac{1}{2} \cdot \frac{p(p+1)}{(p-1)^3} - \frac{1}{2} \cdot \frac{p}{(p-1)^2} =$$

$$= \frac{1}{2} \cdot \frac{p(p+1) - p(p-1)}{(p-1)^3} = \frac{p}{(p-1)^3}. \tag{68}$$

$$6. \ \{a_n\} = \frac{n(n-1)(n-2)}{3!}.$$

In view of the fact that

$$p\left\{\frac{n(n-1)(n-2)}{3!}\right\} = \left\{\frac{(n+1)n(n-1)}{3!}\right\}$$

and

$$\left\{\frac{(n+1)n(n-1)}{3!}\right\} - \left\{\frac{n(n-1)(n-2)}{3!}\right\} = \left\{\frac{n(n-1)}{2!}\right\},$$

we shall obtain

$$\left\{\frac{n(n-1)(n-2)}{3!}\right\} = \frac{1}{p-1}\left\{\frac{n(n-1)}{2!}\right\} = \frac{p}{(p-1)^4}. \tag{69}$$

$$7. \ \{a_n\} = \{c^{an}\}.$$

By virtue of Formula (24), we have

$$\{c^{\alpha(n+1)}\} = p(\{c^{\alpha n}\} - 1),$$

whence

$$\{c^{\alpha n}\} c^{\alpha} = p\{c^{\alpha n}\} - p,$$

$$\{c^{\alpha n}\} = \frac{p}{p - c^{\alpha}}; \tag{70}$$

and

$$\{(-c)^{\alpha n}\} = \frac{p}{p + c^{\alpha}}.$$

1.7. Examples of solving difference equations

Example 1

Let

$$x_n + 2\hat{x}_{n+1} = \hat{1}; \quad n \geqslant 0$$

be a linear difference equation

Determine $\hat{X}_n$ with the assumption that $X_o = 0$.

Solution

Making use of Formula (24), we write

$$\hat{x}_n + 2(p\hat{x}_n - px_0) = \hat{1}$$

hence after taking into account that $X_O = 0$, we shall obtain

$$\hat{x}_n = \frac{\hat{1}}{2p+1}.$$

Then, taking into consideration the relation (64)

$$\hat{1} = \{1\} = \frac{p}{p-1};$$

we have

$$\hat{x}_n = \frac{p}{(p-1)(2p+1)} = \frac{\frac{1}{3}}{p-1} + \frac{\frac{1}{3}}{2p-1} =$$

$$= \frac{1}{3}\{1_{n-1}\} + \frac{1}{6}\left\{\left(-\frac{1}{2}\right)^{n-1}\right\} = \frac{1}{3}(0, 1, 1, 1, \ldots) + \frac{1}{6}\left(0, 1, -\frac{1}{2}, \frac{1}{4}, \ldots\right).$$

Example 2

Let

$$\hat{x}_{n+4} + 2\hat{x}_{n+3} + 3\hat{x}_{n+2} + 2\hat{x}_{n+1} + \hat{x}_n = 0; \quad n \geqslant 0$$

be a linear differential equation.

Determine $\hat{x}_n$, the following boundary conditions being assumed

$$x_0 = x_1 = x_3 = 0, \quad x_2 = -1.$$

Solution

Using the formula (24) we write

$$\hat{x}_n(p^4 + 2p^3 + 3p^2 + 2p + 1) = -p^2 - 2p,$$

or

$$\hat{x}_n = \frac{-p^2 - 2p}{(p^2+p+1)^2} = \frac{-p^2 - 2p}{(p-p_1)^2(p-p_2)^2} = \frac{K_1}{(p-p_1)^2} + \frac{K_2}{p-p_1} +$$

$$+ \frac{K_3}{(p-p_2)^2} + \frac{K_4}{p-p_2},$$

where

$$p_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2\pi}{3}},$$

$$p_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{i\frac{2\pi}{3}},$$

and

$$K_1 = -\frac{p_1^2 + 2p_1}{(p_1 - p_2)^2} = -\frac{1}{2} = i\frac{\sqrt{3}}{6},$$

$$K_3 = -\frac{p_2^2 + 2p_2}{(p_1 - p_2)^2} = -\frac{1}{2} - i\frac{\sqrt{3}}{6},$$

$$K_2 = -K_4 = \frac{2}{(p_1 - p_2)^3}(p_1 + p_2 + p_1 p_2) = 0,$$

hence, in accordance with Formula (12) in the table of numerical operators (p. 96).

$$\hat{x}_n = \frac{K_1}{(p - p_1)^2} + \frac{K_3}{(p - p_2)^2} = K_1\{(n-1)p_1^{n-2}\} + K_3\{(n-1)p_2^{n-2}\} =$$

$$= \frac{2}{\sqrt{3}}\left\{(n-1)\sin\frac{2\pi}{3}n\right\}.$$

Example 3

Let

$$\hat{x}_{n+1}^2 + 2\hat{x}_{n+1} - 3 = 0, \qquad n \geqslant 0$$

be a "nonlinear" difference equation.

Determine $\hat{X}_n$ with the assumption that $X_O = 0$.

Solution

Using Formula (24), we writc $p^2\hat{x}_n^2 + 2p\hat{x}_n - 3 = 0$, whence

$$\hat{x}_{n1,2} = \frac{-2p \pm \sqrt{4p^2 + 12p^2}}{2p^2} = -q \pm 2q,$$

or

$$\hat{x}_{n1} = -q + 2q = q = 0, 1, 0, 0, \ldots,$$

$$\hat{x}_{n2} = -q - 2q = -3q = 0, -3, 0, 0, \ldots.$$

2. APPLICATION OF THE NUMERICAL OPERATOR METHOD IN ELECTRICAL ENGINEERING

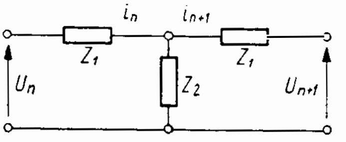
2.1. Application of the numerical operator method to the determination of the energy relations in ladder networks

2.1.1. Example of a ladder line

By means of the numerical operator method, we can solve several practical problems in electrical engineering, theoretical physics, statistics, etc.

As an example, we shall determine, by means of this method, the energy relations in a ladder line consisting of four-poles of the type T.

Fig. 2. Link of a chain of symmetric four-poles of the type T



On the basis of the Kirchhoff law, we can set the following equations

$$\begin{aligned} (z_1 + z_2)\, i_n - z_2\, i_{n+1} - u_n &= 0\,, \\ z_2\, i_n - (z_1 + z_2)\, i_{n+1} - u_{n+1} &= 0\,. \end{aligned} \tag{71}$$

Formulae (71) represent a set of two homogeneous difference equations of the first order.

Our task is to determine the quantities i_n and u_n for $n = 0, 1, 2, 3 \ldots$ This task is equivalent to finding two operators $\{i_n\}$ and $\{u_n\}$ satisfying

$$\begin{aligned} (z_1 + z_2)\,\{i_n\} - z_2\,\{i_{n+1}\} - \{u_n\} &= 0, \\ z_2\,\{i_n\} - (z_1 + z_2)\,\{i_{n+1}\} - \{u_{n+1}\} &= 0\,. \end{aligned} \tag{72}$$

Taking now into consideration the law of the displacement operation (24), we shall obtain

$$\{i_{n+1}\} = p\,(\{i_n\} - i_0)$$

and (73)

$$\{u_{n+1}\} = p\,(\{u_n\} - u_0)\,,$$

where i_O, u_O = the boundary values of $\{i_n\}$, $\{u_n\}$ for $n = 0$.

Taking into account (73) in (72) and solving the set of algebraic equations with respect to $\{i_n\}$ and $\{u_n\}$, we shall arrive at

$$\{i_n\} = \frac{p\left[p - \left(1 + \frac{z_1}{z_2}\right)\right] i_0 - p\,\frac{u_0}{z_2}}{p^2 - 2\left(1 + \frac{z_1}{z_2}\right)p + 1}\,,$$

$$\{u_n\} = \frac{p\left[p - \left(1 + \frac{z_1}{z_2}\right)\right] u_0 - z_2\, p\left[\left(1 + \frac{z_1}{z_2}\right)^2 - 1\right] i_0}{p^2 - 2\left(1 + \frac{z_1}{z_2}\right)p + 1}$$

if we denote

$$1 + \frac{z_1}{z_2} = \cosh \beta\,,$$

whence

$$\left(1 + \frac{z_1}{z_2}\right)^2 - 1 = \sinh^2 \beta,$$

then we shall obtain

$$\{i_n\} = \frac{p\,[p - \cosh \beta]\, i_0 - p\,\dfrac{u_0}{z_2}}{p^2 - 2p \cosh \beta + 1},$$

$$\{u_n\} = \frac{p\,[p - \cosh \beta]\, u_0 - p z_2 \sinh^2 \beta i_0}{p^2 - 2p \cosh \beta + 1}.$$

Now, if we take into consideration Formulae (23) and (24) in the table of numerical operators (p. 96), we shall obtain

$$\{i_n\} = i_0 \{\cosh \beta n\} - \frac{u_0}{z_2 \sinh \beta} \{\sinh \beta n\},$$

$$\{u_n\} = u_0 \{\cosh \beta n\} - i_0\, z_2 \sinh \beta \{\sinh \beta n\}$$

or

$$\begin{aligned} \{i_n\} &= i_0 \{\cosh \beta n\} - u_0 Y \{\sinh \beta n\}, \\ \{u_n\} &= u_0 \{\cosh \beta n\} - i_0 Z \{\sinh \beta n\}, \end{aligned} \qquad (74)$$

where

$$Y = \frac{1}{Z} = \frac{1}{z_2 \sinh \beta}.$$

The expressions (74) constitute a general solution of the problem under consideration and contain the constants i_O, u_O which are to be determined from the boundary conditions.

Let us now consider a particular case in which the line is short circuited. Assume that the N-th four-pole is short-circuited — that is

$$u_N = 0. \qquad (75)$$

Taking into account (75) in Equations (74), we shall obtain

$$0 = u_0 \cosh \beta N - i_0 Z \sinh \beta N,$$

whence

$$i_0 = \frac{u_0}{Z} \cdot \frac{\cosh \beta N}{\sinh \beta n} = u_0 Y \coth \beta N.$$

Substituting i_O into Equations (74), we shall obtain

$$\begin{aligned} \{i_n\} &= u_0 Y [\coth \beta N \{\cosh \beta n\} - \{\sinh \beta n\}] = \frac{u_0 Y}{\sinh \beta n} \{\cosh \beta (N - n)\}, \\ \{u_n\} &= u_0 [\{\cosh \beta n\} - \coth \beta N \{\sinh \beta n\}] = \frac{u_0}{\sinh \beta N} \{\sinh \beta (N - n)\} \end{aligned} \qquad (76)$$

if the line consisting of N four-poles is in an open-circuit condition — that is, if

$$i_N = 0,$$

then we shall find

$$i_0 = u_0 Y \frac{\sinh \beta N}{\cosh \beta N} = u_0 Y \tanh \beta N .$$

Substituting i_O into Equations (74), we shall arrive at

$$\{i_n\} = u_0 Y [\tanh \beta N \{\cosh \beta n\} - \{\sinh \beta n\}] = \frac{u_0 Y}{\cosh \beta N} \{\sinh \beta (N - n)\},$$

$$\{u_n\} = u_0 [\{\cosh \beta n\} - \tanh \beta N \{\sinh \beta n\}] = \frac{u_0}{\cosh \beta N} \{\cosh \beta (N - n)\}. \qquad (77)$$

Now in view of the fact that

$$\lim_{N\to\infty} \left[\frac{\cosh \beta (N - n)}{\cosh \beta N}\right] = \lim_{N\to\infty} \left[\frac{e^{\beta(N-n)} + e^{-\beta(N-n)}}{e^{\beta N} + e^{-\beta N}}\right] =$$

$$= \lim_{N\to\infty} \left[\frac{e^{-\beta N} + e^{-\beta(2N-n)}}{1 + e^{-2\beta N}}\right] = e^{-\beta n}, \qquad (78)$$

and similarly

$$\lim_{N\to\infty} \left[\frac{\sinh \beta (N - n)}{\cosh \beta N}\right] = e^{-\beta n} \qquad (79)$$

we shall obtain the well-known equation for an infinite network

$$\{i_n\} = u_0 Y \{e^{-\beta n}\}$$
$$\{u_n\} = u_0 \{e^{-\beta n}\} \qquad (80)$$

2.1.2. String of high-voltage insulators

Figure 3 represents a string of high-voltage insulators and shows a simplified equivalent circuit of such a string.

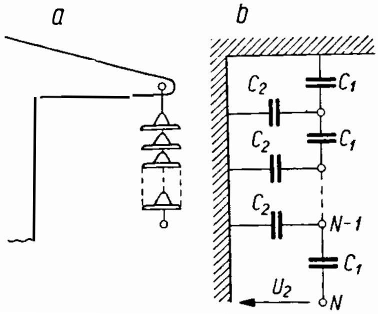


Fig. 3. Chain of high voltage insulators

In accordance with the Kirchoff laws, we can easily establish the following equation

$$j\omega c_1 (u_{n+2} - u_{n+1}) = j\omega c_1 (u_{n+1} - u_n) + i\omega c_2 u_{n+1},$$

whence

$$\{u_{n+2}\} - 2\left(1 + \frac{c_2}{2c_1}\right)\{u_{n+1}\} + \{u_n\} = 0. \tag{81}$$

The expression (81) is a linear difference equation of the second order, and can be solved by means of the numerical operator method.

Taking into consideration Formula (24), we shall find

$$\begin{aligned} \{u_{n+1}\} &= p(\{u_n\} - u_0), \\ \{u_{n+2}\} &= p^2(\{u_n\} - u_0) - pu_1. \end{aligned} \tag{82}$$

Taking into account (82) in Equation (81), and assuming that $u_O = 0$, we shall obtain

$$\left[p^2 - 2\left(1 + \frac{c_2}{2c_1}\right)p + 1\right]\{u_n\} = pu_1,$$

whence

$$\{u_n\} = \frac{pu_1}{p^2 - 2\left(1 + \frac{c_2}{2c_1}\right)p + 1}. \tag{83}$$

After denoting

$$\cosh x = 1 + \frac{c_2}{2c_1},$$

$$\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{c_2}{c_1}\left(1 + \frac{c_2}{4c_1}\right)}, \tag{84}$$

we have

$$\{u_n\} = \frac{p \sinh x}{p^2 - 2p\cosh x + 1} \cdot \frac{u_1}{\sinh x} \tag{85}$$

and after taking into consideration Formula (23) in the table of numerical operators, we obtain

$$\{u_n\} = \frac{u_1}{\sinh x}\{\sinh xn\}. \tag{86}$$

The quantity u_1 is found from the boundary conditions; namely, we assume that

$$u_N = U_z = \text{voltage of the line with respect to the earth} \tag{87}$$

which assumption together with (86) yields

$$u_1 = U_z \frac{\sinh x}{\sinh xN}. \tag{88}$$

Taking into consideration the expression obtained from u_1 in Equation (86), we shall obtain

$$\{u_n\} = \frac{U_z}{\sinh xN} \{\sinh xn\}. \tag{89}$$

If $\frac{c_2}{2c_1}$ is small, then $\sinh xn \approx xn$, $\sinh xN \approx xN$, and

$$\{u_n\} \approx \frac{U_z}{N} \{n\} \tag{90}$$

and then voltages occur on each insulator.

PART II

STABILITY OF SAMPLED-DATA CONTROL SYSTEMS

1. INTRODUCTION

Control systems may according to their properties be classified into control systems with continuous action, and control systems with non-continuous action.

Among control systems with continuous action, we include systems in which the signal correcting the controlled quantity is a continuous function, and the control circuit is always closed. Among systems with non-continuous action, we include control systems which do not satisfy the above requirement.

Systems of important type are what are called "sampled-data control systems" — namely, those in which the control circuits are closed periodically. In what follows, we shall confine ourselves to the investigation of sampled-data control systems of a suitable type which is widely used in practice. This system can always be represented in the form of a system with a sampled-data feedback (Fig. 6).

The signal correcting the controlled quantity is not a continuous function in the systems under consideration, but "consists of" rectangular pulses having an identical width γT and occurring in equal time intervals T. Assume that the heights of these pulses at the instant of their appearance are proportional to the value of the error function $\delta(t) = x_2(t) - x_0(t)$ — that is, to the difference between the controlled quantity and the given quantity.

Sampled-data systems have found wide application in practice, since they have proved to have several advantages over systems with continuous action.

A basic, elementary condition of the correct operation of control systems, and thus in particular also of sampled-data control systems, is their stability. It is known that the stability of a dynamical system consists in the property of its pulse response (the function of the input signal actuated by the excitation in the form of the Dirac function) being a "decaying" function, and thus satisfying the condition

$$\lim_{t\to\infty} k(t) = 0. \tag{91}$$

In the engineering literature known to the present author, the methods of investigating the stability of sampled-data control systems are principally based on examining the frequency characteristics of the system [4]. Since for sampled-data systems it is almost impossible to determine frequency characteristics experimentally, the methods mentioned above are in practice inadequate in these cases in which the analysis af a system must be made in terms of the data obtained from measurments.

The method presented below makes it possible to determine the stability of a sampled-data control system directly from knowledge of the frequency characteristic of an "open system". Moreover, an analysis of typical frequency characterisics is made, as a result of which we arrive at the diagrams determining the "stability zones" of the sampled--data control systems under consideration. The results enable us in several instances to effect the investigation of the stability almost immediately in terms of the plot of the time characteristic (response to a unit-step excitation) of an open control system, which can easily be found experimentally. Noted that in the case of systems with continuous action, such a procedure is often used [9].

The author has shown in [15] that the numerical operator method can be applied in analysis of sampled-data systems. The method of investigating sampled-data systems, based on the numerical operator method, is more general and simpler than that presented in [4] by Y. Z. Tsypkin; the latter uses the transformation

$$F(q) = \sum_{n=0}^{\infty} f(n)\, e^{-qn}, \tag{92}$$

where $f(n)$ is a function of a variable integer n ($n = 0, 1, 2, \ldots$) and $F(q)$ is a function of a complex variable q.

It is obvious that the transformation (92) may be used only for functions which ensure the convergence of the series

$$\sum_{n=0}^{\infty} f(n)\, e^{-qn}. \tag{93}$$

It is then impossible to transform the functions

$$\begin{aligned} f(n) &= n\,!, \\ f(n) &= n^2\,!, \\ f(n) &= e^{n^2}, \text{ etc.} \end{aligned} \tag{94}$$

The numerical operator method does not impose any such restrictions on the class of the functions $f(n)$.

It was proved in the paper [15] that the "transmission properties" of a sampled-data system can always be determined by means of what is called "equation of the sampled-data system"

$$\{x_2(n, \varepsilon)\} = K_i(p, \varepsilon)\{x_1(n)\}, \qquad 0 \leqslant \varepsilon < 1, \tag{95}$$

where

$x_2(n, \varepsilon)$ — is a function determining the signal on the output of the sampled-data system,

$x_1(n)$ — is a sequence of the values of the input signal $x_1(t)$ corresponding to the moments of generating pulses,

$K_i(p, \varepsilon)$ — is the "transfer function" of the system,

$K_i(p, \varepsilon)$ — is determined by one of two different formulae

$K_I(p, \varepsilon)$ or $K_{II}(p, \varepsilon)$, depending on whether the real parameter ε is contained in the interval $[0, \gamma)$, or in the interval $[\gamma, 1)$, where γ is the quotient

$$\gamma = \frac{\text{width of pulses}}{\text{period}} \tag{96}$$

A block diagram of the sampled-data system is shown in Fig. 4. This system consists of two elements in a series connection — namely, what are called the "sampler" and the "linear element".

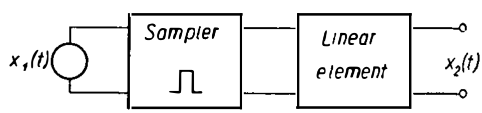


Fig. 4. Diagram of a sampled-data system

A sampler may be the signalling key, the electromechanical relay, the electronic relay, etc. Its task is to perform the transformation of the continuous function of the input signal $x_1(t)$ into a signal in the form of rectangular pulses with the width γT and the period T. The heights of these pulses at the moment of their appearance are not proportional to the value of the signal $x_1(t)$, and thus they are "modulated" by the function $x_1(t)$.

A linear element may be any linear dynamical system capable of transmitting the signal.

Sampled-data systems may be connected in a manner identically to that used for "ordinary" dynamical systems. A series connection, and a connection in a feedback system (Fig. 5, 6, 7) are also possible.

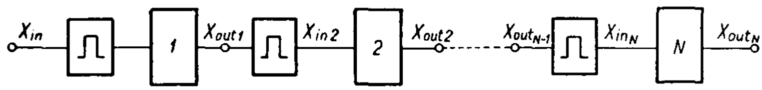


Fig. 5. Series connection of sampled-data systems

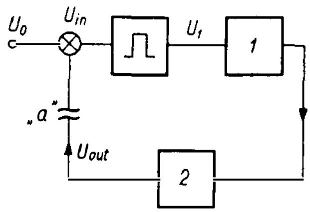


Fig. 6. Feedback sampled--data system

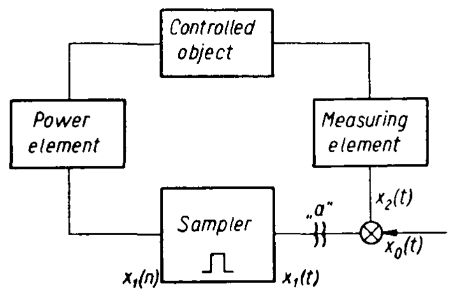


Fig. 7. Basic diagram of a sampled-data systems

It can easily be shown that in a sampled-data feedback system (Fig. 7), the following relation between the signal X_2 and X_0 holds

$$\{x_2(n, \varepsilon)\} = \frac{K_i(p, \varepsilon)}{1 + K_l(p, 0)} \{x_0(n)\}; \qquad 0 \leqslant \varepsilon < 1, \tag{97}$$

where $K_i(p, \varepsilon)$ is the transfer function of the sampled-data system obtained by the "opening" of the feedback branch in the function "a" — that is, before the sampler. The deduction of the above formula is given in the annotations.

The relation (97) will be referred to as the "equation of a feedback sampled-data system", and the expression

$${}_sK_i(p, \varepsilon) = \frac{K_i(p, \varepsilon)}{1 + K_l(p, 0)} \tag{98}$$

will be referred to as the "transfer function" of the "characteristic function" of a feedback sampled-data system.

Each sampled-data control system under consideration may be represented as a feedback sampled-data system, and in this respect the dynamical properties can always be determined by Formula (97).

In view of the manner of the analysis performed, the author finds it reasonable to distinguish what are called "static" and "astatic" systems. Such a classification of control systems, of course, is not indispensable.

2. DETERMINATION OF THE STABILITY OF A SAMPLED-DATA CONTROL SYSTEM FROM THE TIME CHARACTERISTIC OF THE OPEN SYSTEM

Let us now discuss a feedback sampled-data system (Fig. 7). If the feedback circuit is broken before the sampler (at the point a), then we shall obtain an ordinary sampled-data system the theory of which was presented in the paper [15].

Let us write the equation of the open system at the point (a)

$$\{x_2(n, \varepsilon)\} = K_i(p, \varepsilon)\{x_1(n)\}. \tag{99}$$

Assume now that on its input — that is, at the point (a) — an excitation was applied in the form of a unit-step function. Then, on the output of the sampler there will appear a sequence of rectangular pulses having identical unit heights; thus we shall write

$$\{x_1(n)\} = \{1\} = \frac{p}{p-1}, \tag{100}$$

Hence the signal $x_2(t)$ may be defined by the formula

$$\{x_2(n, \varepsilon)\} = K_i(p, \varepsilon)\frac{p}{p-1}. \tag{101}$$

Since a numerical operator can always be represented in the form of the power series

$$\{a_n\} = \sum_{n=0}^{\infty} a_n p^{-n}, \tag{102}$$

we shall then obtain

$$K_i(p, \varepsilon) = \frac{p-1}{p}\sum_{n=0}^{\infty} x_2(n, \varepsilon)p^{-n} = (1 - p^{-1})\sum_{n=0}^{\infty} x_2(n, \varepsilon)p^{n} =$$

$$= x_2(0, \varepsilon) + \sum_{n=1}^{\infty}[x_2(n, \varepsilon) - x_2(n-1, \varepsilon)]p^{-n}. \tag{103}$$

From Formula (103), we can determine the frequency characteristic of the sampled-data system. If — in accordance with Tsypkin — we determine the frequency characteristic $B(j\bar{\omega}, \varepsilon)$* of the sampled-data

---

* $\bar{\omega}$ detones the dimensionless angular frequency; $\bar{\omega} = \omega T$ where T, is the period of the occurrence of pulses.

system by the formula

$$B(j\bar{\omega}, \varepsilon) = \frac{x_{2\,ust}(n, \varepsilon)}{x_1(n)} = \frac{x_{2\,ust}(u, \varepsilon)}{ce^{j\bar{\omega}n}}, \qquad (104)$$

where x_2 *steady* (n, ε) is a steady component of the output signal $x_2(n, \varepsilon)$ and

$$x_1(n) = ce^{j\bar{\omega}n} \qquad (105)$$

is the excitation signal, then we can easily explain the following property. The frequency characteristic $B(j\bar{\omega}, \varepsilon)$ of the sampled-data system is equal to the expression $K_i(e^{j\bar{\omega}}, \varepsilon)$ obtained by the substitution in the pulse transfer function $K_i(p, \varepsilon)$ of the system the function $e^{j\bar{\omega}}$ for the displacement operator p;

$$B(j\bar{\omega}, \varepsilon) = K_i(e^{j\bar{\omega}}, \varepsilon). \qquad (106)$$

The above property can be accounted for in terms of the equation of the sampled-data system, if we assume that $x_1(n) = ce^{j\omega n}$.

After elementary transformations, we obtain (141). This property follows also from Formula (2-116) given in the paper [4] by Tsypkin, and from the isomorphism which holds between the numerical operators and the summation Laplace transformation.

According to (104), the frequency characteristic of a sampled data system can be determined from knowledge of $x_2(n, \varepsilon)$

$$B(j\bar{\omega}, \varepsilon) = K_i(e^{j\bar{\omega}}, \varepsilon) = x_2(0, \varepsilon) + \sum_{n=1}^{\infty} [x_2(n, \varepsilon) - x_2(n-1, \varepsilon)] e^{-j\bar{\omega}n},$$

where $x_2(n, \varepsilon)$ is a function of the output signal actuated by the unit-step excitation

$$x_1(t) = \mathbf{1}(t).$$

If, in turn, we assume that the (open) sampled-data system is stable, since

$$\lim_{n \to \infty} [x_2(n, \varepsilon) - x_2(n-1, \varepsilon)] = 0^*, \qquad (107)$$

then in Formula (103) we can disregard all the terms having sufficiently great indices ($n > N$), for which

$$|x_2(n, \varepsilon) - x_2(n-1, \varepsilon)| < \delta, \qquad (108)$$

where for δ we may in practice accept for example $0.1k$ or $0.05k$ (k is the

* This formula is a necessary but not sufficient condition for the stability of a sampled-data system.

resulting amplification in the feedback loop). We shall then obtain the approximate formula

$$K_i(e^{j\bar{\omega}}, \varepsilon) \approx x_2(0, \varepsilon) + \sum_{n=1}^{N} [x_2(n, \varepsilon) - x_2(n-1, \varepsilon)] e^{-j\bar{\omega}n}, \tag{109}$$

which is sufficiently accurate for determining the stability of a closed sampled-data system.

In accordance with the results obtained by Tsypkin [4] (pp. 148—155), the stability of the system is determined from what follows

Theorem. A feedback sampled-data system is stable if and only if the Nyquist diagram $K(e^{j\bar{\omega}},0)$ of the open system, with a change in ω from $-\pi$ to $+\pi$, does not contain the point -1, $j0$.

In view of the above considerations, the method of investigating the stability of a sampled-data control system in terms of the time characteristic $x_2(n, \varepsilon)$ of the open system reduces to

1° determining on the basis of Formula (109) the function $K(e^{j\omega}, 0)$ from knowledge of the time characteristic $x_2(n, \varepsilon)$ of the open system,

2° plotting the Nyquist diagram.

The stability of the system is then found on the basis of the theorem given above.

The Nyquist diagram $K(e^{j\omega},0)$ can easily be determined by means of calculating its components

$$\begin{aligned} \mathcal{R}e\, K(e^{j\bar{\omega}}, 0) &= x_2(0, 0) + \sum_{n=1}^{N} [x_2(n, 0) - x_2(n-1, 0)] \cos n\bar{\omega}, \\ \mathcal{I}m\, K(e^{j\bar{\omega}}, 0) &= - \sum_{n=1}^{N} [x_2(n, 0) - x_2(n-1, 0)] \sin n\bar{\omega}. \end{aligned} \tag{110}$$

From the above considerations we may draw the interesiting conclusion that the stability of a sampled-data control system is dependent only and exclusively on the waveform of the time characteristic $x_2(\bar{t})$ at the moments $t = \frac{\bar{t}}{T} = (n = 0, 1, 2 \ldots)$ — that is, the waveform of $x_2(n, 0)$. The behavior of the $x_2(\bar{t})$ characteristic between the moments $\bar{t} = n$ has absolutely on influence on the stability of the system. This fact becomes clear if we bear in mind that owing to the presence of the sampler in the feedback branch, the control system is closed only in the moments $t = n$.

The waveform of the time characteristic $x_2(n, \varepsilon)$ can of course be determined experimentally or may be calculated by the analytical method. It should de emphasized that the method presented is suitable for the investigation of systems which are stable after opening the feedback branch — that is to say, to the investigation of systems containing no elements with what is called the astatic characteristic.

We shall prove below that this restriction can be avoided in a relatively simple manner. Namely, let us observe that for an (open) sampled-data system with an astatic characteristic and with only one integrating element, the following condition must be satisfied

$$\lim_{n\to\infty} \Delta^2 x_2(n-1,\varepsilon) = 0; \qquad 0 \leqslant \varepsilon < 1, \tag{111}$$

where

$$\begin{aligned}\Delta^2 x_2(n-1,\varepsilon) &= \Delta[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)] = \\ &= x_2(n+1,\varepsilon) - 2x_2(n,\varepsilon) + x_2(n-1,\varepsilon),\end{aligned} \tag{112}$$

and $x_2(n, \varepsilon)$ is the response of the sampled-data system under consideration to unit-step excitation.

If the formula determining $K_i(e^{j\bar{\omega}}, 0)$ is transformed, so that under the sign of the sum appear the terms of the form $\Delta^2 x_2(n-1, 0)e^{j\omega_n}$, it will be possible, as in the approximate formula, to have a finite number of components in the series. Let us now pursue the following reasoning.

Multiply both sides of Formula (103) by $p-1$; we shall obtain

$$(p-1)K_i(p,\varepsilon) = (p-1)x_2(0,\varepsilon) + (p-1)\sum_{n=1}^{\infty}[x_2(n,\varepsilon) - x_2(p-1,\varepsilon)]p^{-n}, \tag{113}$$

hence, after elementary transformations

$$(p-1)K_i(p,\varepsilon) = (p-1)x_2(0,\varepsilon) + \Delta x_2(0,\varepsilon) + \sum_{n=1}^{\infty}\Delta[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)]p^{-n},$$

or

$$(p-1)K_i(p,\varepsilon) = (p-1)x_2(0,\varepsilon) + \Delta x_2(0,\varepsilon) + \sum_{n=1}^{\infty}\Delta^2 x_2(n-1,\varepsilon)p^{-n}, \tag{114}$$

where

$$\begin{aligned}\Delta^2 x_2(n-1,\varepsilon) &= \Delta[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)] = \\ &= x_2(n+1,\varepsilon) - 2x_2(n,\varepsilon) + x_2(n-1,\varepsilon)\end{aligned} \tag{115}$$

and

$$\Delta x_2(0,\varepsilon) = x_2(1,\varepsilon) - x_2(0,\varepsilon). \tag{116}$$

The frequency characteristic $K(e^{j\bar{\omega}}, 0)$ of an "open" sampled-data control system may then be expressed by the formula

$$K(e^{j\bar{\omega}}, 0) = \frac{1}{e^{j\bar{\omega}} - 1} \sum_{n=1}^{\infty} \Delta^2 x_2(n-1, 0) e^{-j\bar{\omega}n} + x_2(0, 0) + \frac{\Delta x_2(0, 0)}{e^{j\bar{\omega}} - 1}. \quad (117)$$

Further, using the relation (111) which is correct for a system with an astatic characteristic, we may in the above formula omit all the terms with the indices $n > N$. Then we shall obtain the approximate formula

$$\boxed{K(e^{j\bar{\omega}}, 0) \approx \frac{1}{e^{j\bar{\omega}} - 1} \sum_{n=1}^{N} \Delta^2 x_2(n-1, 0) e^{-j\bar{\omega}n} + x_2(0, 0) + \frac{\Delta x_2(0, 0)}{e^{j\bar{\omega}} - 1},}$$

We may also deduce formulae determining the components of the complex function $K(e^{j\bar{\omega}}, 0)$. Taking into consideration the obvious realations

$$\mathcal{R}e\left\{\frac{1}{e^{j\omega} - 1}\right\} = -\frac{1}{2}; \qquad \mathcal{I}m\left\{\frac{1}{e^{j\bar{\omega}} - 1}\right\} = -\frac{1}{2}\cot\frac{\bar{\omega}}{2} \quad (119)$$

and utilizing the formulae defining the components of the product of complex numbers, we shall obtain:

$$\mathcal{R}e\, K(e^{j\bar{\omega}}, 0) \approx x_2(0, 0) - \frac{1}{2}\Delta x_2(0, 0) + \quad (120)$$

$$-\frac{1}{2}\sum_{n=1}^{N} \Delta^2 x_2(n-1, 0)\cos\bar{\omega}n - \frac{1}{2}\cot\frac{\bar{\omega}}{2}\sum_{n=1}^{N} \Delta^2 x_2(n-1, 0)\sin\bar{\omega}n,$$

$$\mathcal{I}m\, K(e^{j\bar{\omega}}, 0) \approx \frac{1}{2}\sum_{n=1}^{N} \Delta^2 x_2(n-1, 0)\sin\bar{\omega}n + \quad (121)$$

$$-\frac{1}{2}\cot\frac{\bar{\omega}}{2}\sum_{n=1}^{N} \Delta^2 x_2(n-1, 0)\cos\bar{\omega}n - \frac{1}{2}\cot\frac{\bar{\omega}}{2}\Delta x_2(0, 0).$$

On the basis of the formulae obtained defining $Re\, K(\epsilon^{j\bar{\omega}}, 0)$ and $Im\, K(e^{j\bar{\omega}}, 0)$ it is possible, from knowledge of the time characteristic $x_2(n, 0)$, to determine the plot of the Nyquist diagram of an open sampled--data system containing an integrating element (for example, a servo-motor). Knowing the plot of this diagram in the interval $\bar{\omega}\epsilon[-\pi, \pi]$, we can determine the stability of the control system.

In conclusion, it is worth mentioning that knowledge of the Nyquist diagram $K(e^{j\bar{\omega}}, 0)$ may be useful not only in determining the stability of a control system. From the plot of the function $K(\epsilon^{j\bar{\omega}}, 0)$, we can also anticipate what correcting elements should be used in order to improve the quality of the control process.

3. CONDITIONS FOR THE STABILITY OF A STATIC SAMPLED-DATA CONTROL SYSTEM WITH O TYPICAL CHARACTERISTIC

The method presented in the preceding chapter may be applied to any static or astatic sampled-data control system. In spite of its universality, this method has a certain drawback, since it is necessary to perform preliminary calculations in order to investigate the stability of a system. It will be shown below that in the case in which the time characteristic $x_2(n, 0)$ of an open sampled-data control system is typical, as indicated in Fig. 8a, the stability of the system may be determined directly from the plot of $x_2(n, 0)$ without the necessity to determine the Nyquist diagram $K(e^{j\bar{\omega}}, 0)$. The reasoning is based on the substitution of a simplified characteristic in the form of a broken line in Fig. 8b for the characteristic in the form of a broken line in Fig. 8b for the characteristic from Fig. 8a.

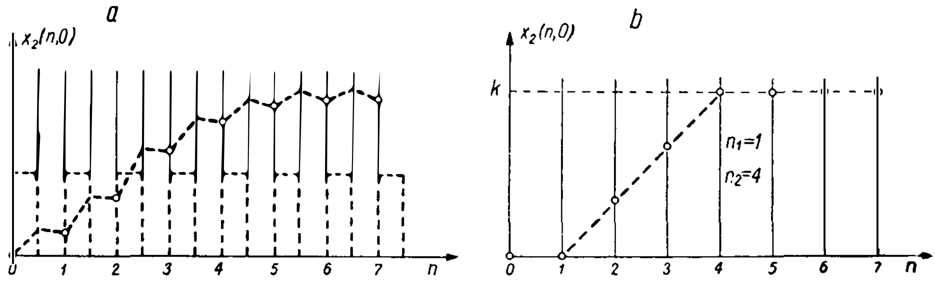


Fig. 8. Approximation of the time charac teristic of a sampled-data system by means of the broken line

The simplified characteristic can be written analytically in the following manner

$$x_2(n, 0) = \begin{cases} 0; \; n < n_1 \\ \dfrac{k}{n_2 - n_1}(n - n_1); \; n_1 \leqslant n < n_2 \\ k; \; n \geqslant n_2 \end{cases} \tag{122}$$

Since the input signal $x_2(t)$ is actuated by a unit-step excitation $\mathbf{1}(t)$, then in accordance with the formula presented above, the coefficient k is equal to the "resultant" amplification in the feedback loop.

Using the numerical operator method, we may express the simplified characteristic $x_2(n, 0)$ by the formula

$$\{x_2(n, 0)\} = \frac{k}{n_2 - n_1} \cdot \frac{p}{(p-1)^2} [p^{-n_1} - p^{-n_2}], \tag{123}$$

where p is the displacement operator

Since we assumed that a unit-step excitation was applied to the input of the system — that is,

$$\{x_1(n)\} = \{1\} = \frac{p}{p-1}, \tag{124}$$

then

$$K(p, 0) = \frac{\{x_2(n, 0)\}}{\{x_1(n, 0)\}} = \frac{k}{n_2 - n_1} \frac{p^{-n_1} - p^{-n_2}}{p-1}. \tag{125}$$

Substituting into the above formula for the place of the displacement operator p the function $e^{j\bar{\omega}}$, we shall obtain the expression $K(e^{j\bar{\omega}}, 0)$ which determines the frequency characteristic of the open control system

$$K(e^{j\bar{\omega}}, 0) = \frac{k}{n_2 - n_1} \cdot \frac{e^{-j\bar{\omega}n_1} - e^{-j\bar{\omega}n_2}}{e^{j\omega} - 1}. \tag{126}$$

By virtue of the theorem cited in the preceding chapter, the stability limit of the system is determined from the conditions

$$\begin{aligned} &\text{a)}\ \Re e\, K(e^{j\bar{\omega}}, 0) = -1, \\ &\text{b)}\ \Im m\, K(e^{j\bar{\omega}}, 0) = 0 \end{aligned} \tag{127}$$

For this purpose we shall investigate the roots of the equation

$$K(e^{j\bar{\omega}}, 0) + 1 = 0. \tag{128}$$

Taking into consideration in the above equation the expression (126), we shall arrive at

$$k(e^{-j\omega n_1} - e^{-j\bar{\omega}n_2}) + (n_2 - n_1)(e^{j\bar{\omega}} - 1) = 0, \tag{129}$$

and hence after performing considerable elementary trigonometric transformations, we shall obtain the following relations (c) and (d)

$$\begin{aligned} &\text{c)}\ 2k \sin \bar{\omega}\alpha \sin \bar{\omega}\beta - (n_2 - n_1)(\cos \bar{\omega} - 1) = 0, \\ &\text{d)}\ 2k \cos \bar{\omega}\alpha \sin \bar{\omega}\beta - (n_2 - n_1)\sin \bar{\omega} = 0, \end{aligned} \tag{130}$$

where

$$\alpha = \frac{n_1 + n_2}{2}, \qquad \beta = \frac{n_1 - n_2}{2}.$$

The above relations will be treated as a set of two equations with two unknowns: $\bar{\omega}$ and k. Dividing equation (c) by (d), we shall find the unknown $\bar{\omega}$:

$$\tan \bar{\omega}\alpha = \frac{\cos \bar{\omega} - 1}{\sin \bar{\omega}} = -\tan \frac{\bar{\omega}}{2}, \tag{131}$$

whence

$$\bar{\omega}\alpha = -\frac{\bar{\omega}}{2} + (\nu + 1)\pi; \qquad \nu = 0, \pm 1, \pm 2, \dots$$

$$\bar{\omega} = \bar{\omega}_\nu = \frac{(\nu + 1)\,2\pi}{2 + n_1 + n_2}. \tag{132}$$

Further, for example from equation (d), we shall determine the unknown k

$$k = k_\nu = (n_2 - n_1)\frac{\sin \bar{\omega}_\nu}{2\cos \bar{\omega}_\nu\,\alpha \sin \bar{\omega}_\nu\,\beta} =$$

$$= (n_2 - n_1)\frac{\sin \dfrac{(\nu + 1)\,2\pi}{1 + n_1 + n_2}}{2\cos \dfrac{n_1 + n_2}{1 + n_1 + n_2}(\nu + 1)\pi \sin \dfrac{n_1 - n_2}{1 + n_1 + n_2}(\nu + 1)\pi}. \tag{133}$$

This formula can be simplified. Namely, if we take into consideration the obvious relation

$$\frac{n_1 + n_2}{1 + n_1 + n_2} = 1 - \frac{1}{1 + n_1 + n_2}, \tag{134}$$

whence

$$\cos \frac{n_1 + n_2}{1 + n_1 + n_2}(\nu + 1)\pi = (-1)^{\nu+1}\cos \frac{(\nu + 1)\pi}{1 + n_1 + n_2}, \tag{135}$$

and, further, if we introduce the transformation

$$\sin \frac{(\nu + 1)\,2\pi}{1 + n_1 + n_2} = 2\sin \frac{(\nu + 1)\pi}{1 + n_1 + n_2}\cos \frac{(\nu + 1)\pi}{1 + n_1 + n_2}, \tag{136}$$

then we shall obtain a simple defining k_ν;

$$k_\nu = (-1)^\nu (n_2 - n_1)\frac{\sin \dfrac{(\nu + 1)\pi}{1 + n_1 + n_2}}{\sin \dfrac{n_2 - n_1}{1 + n_1 + n_2}(\nu + 1)\pi}. \tag{137}$$

This formula is ambiguous and determines the stability conditions for the particular components of the control process. Since we are considering a system with a negative feedback (which is indispensable in the case of every control system), of interest to us in the formula obtained are only those values of the number ν, which yield positive values of the coefficient k , with n_1 and n_2 as assumed. Thus, for example, we should not take into consideration those values of ν, for which the expression

$$\frac{\nu + 1}{1 + n_1 + n_2} \tag{138}$$

is an integral number. For we can verify that then $k_\nu = -1$.

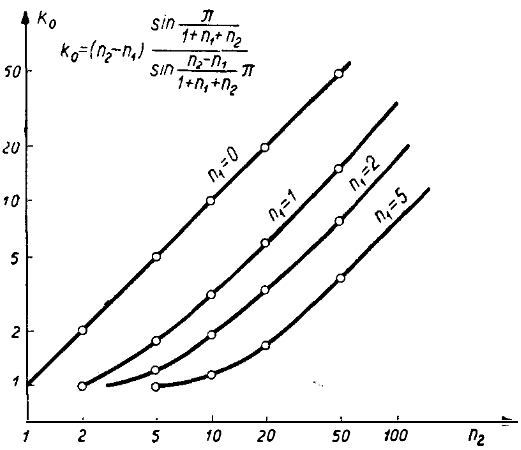
For the fundamental component k_0, we shall obtain the formula

$$k_0 = (n_2 - n_1) \frac{\sin \frac{\pi}{1 + n_1 + n_2}}{\sin \frac{n_2 - n_1}{1 + n_1 + n_2} \pi}. \tag{139}$$

It can be shown that this formula gives the lowest value of the coefficient $k_\nu > 0$. In view of the above, Formula (139) determines the greatest admissible amplification in the feedback loop. For the amplification $k > k_0$, the system is unstable. Figure 9 shows the plots of the function

$$k_0 = f(n_1, n_2), \tag{140}$$

where n_1 is treated as a parameter and n_2 — as an independent variable. Making use of these plots we can easily determine the stability of a static sampled-data control system from the knowledge of the time characteristic $x_2(n, 0)$ of the open system. Note that k_0 is not a function of the ratio of the "delay" time to the "steady-state" time $\frac{n_1}{n_2}$. From the plot of the function $k_0 = f(n_1, n_2)$, it follows that, with n_1 as assumed the



ig. 9. Stability limits of a static sampled--data control system

greater is n_2, that is the slower the time characteristic of the open system increases, the greater the amplification is possible to apply. Of course, we are able to influence the speed of the increase of the time characteristic

$x_2(n, 0)$ by changing the width γ of the rectangular pulses generated by the sampler.

The method presented for determining stability does not require any preliminary calculations.

4. STABILITY CONDITIONS FOR AN ASTATIC SAMPLED-DATA CONTROL SYSTEM WITH A TYPICAL CHARAKTERISTIC

Below, we shall discuss the stability of an astatic sampled-data control system. The analysis performed is based on the assumption that the waveform of the time characteristic $x_2(n, 0)$ (response to a unit-step excitation) has the shape indicated in Fig. 10a. Such a characteristic can be obtained experimentally or can be determined in a simple manner by the analytical graphical method.

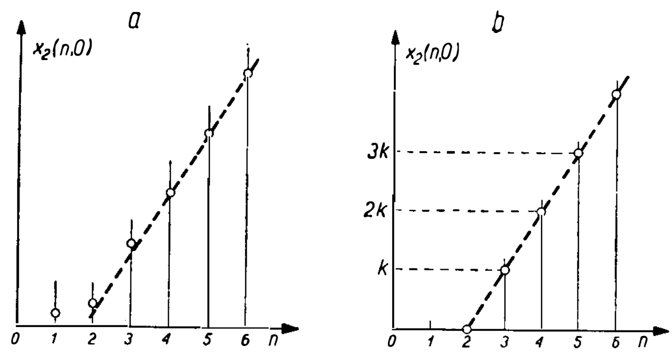


Fig. 10. Approximation of the time characteristic of an astatic sampled-data system by means of the broken line

It should be noted that, as for static systems, the stability of a system depends only on the plot of the characteristic $x_2(n, \varepsilon)$ at the moments $n = 0, 1, 2, \ldots$ and $\varepsilon = 0$, that is, the plot of the characteristic $x_2(n, 0)$. The behavior of the characteristic $x_2(n, \varepsilon)$ between the points $n = 0, 1, 2, \ldots$ has absolutely no bearing on the stability of the system. Thus the method presented may be used for a relatively large class of astatic sampled-data control systems encountered in practice.

In our considerations, we shall replace the characteristic $x_2(n, 0)$ (Fig. 10a) by a simplified characteristic in the form of a straight line (Fig 10b); we shall then assume that

$$\{x_2(n, 0)\} = \begin{cases} 0; & n < n_0 \\ k(n - n_0); & n \geqslant n_0 . \end{cases} \tag{141}$$

Since we have

$$\{n\} = \frac{p}{(p-1)^2}, \tag{142}$$

there will be

$$\{x_2(n, 0)\} = k\frac{p}{(p-1)^2}p^{-n_0}$$

and

$$K(p, 0) = \frac{p-1}{p}\{x_2(n, 0)\} = k\frac{p^{-n_0}}{p-1}. \tag{143}$$

The frequency characteristic of an open control system (corresponding to the moments $\varepsilon = 0$) will be obtained by substituting for the displament operator p the function $e^{j\bar{\omega}}$

$$K(e^{j\bar{\omega}}, 0) = k\frac{e^{-j\bar{\omega}n_0}}{e^{j\bar{\omega}}-1}. \tag{144}$$

Similarly as in the preceding chapter, the stability limit will be determined from the conditions

$$\begin{aligned} &\text{a)}\ \mathcal{R}_e\, K(e^{j\bar{\omega}}, 0) = -1, \\ &\text{b)}\ \mathcal{I}_m\, K(e^{j\bar{\omega}}, 0) = 0. \end{aligned} \tag{145}$$

Taking into consideration (144), we shall obtain

$$\mathcal{R}_e\, K(e^{j\bar{\omega}}, 0) = \mathcal{R}_e\left\{k\frac{e^{-j\bar{\omega}n_0}}{e^{j\bar{\omega}}-1}\right\} =$$

$$= \frac{k}{(\cos\bar{\omega}-1)^2+\sin^2\bar{\omega}}[\cos\bar{\omega}n_0(\cos\bar{\omega}-1)-\sin\bar{\omega}n_0\sin\bar{\omega}],$$

$$\mathcal{I}_m\, K(e^{j\bar{\omega}}, 0) = \mathcal{I}_m\left\{k\frac{e^{-j\bar{\omega}n_0}}{e^{j\bar{\omega}}-1}\right\} =$$

$$= -\frac{k}{(\cos\bar{\omega}-1)^2+\sin^2\bar{\omega}}[\cos\bar{\omega}n_0\cdot\sin\bar{\omega}+\sin\bar{\omega}n_0(\cos\bar{\omega}-1)].$$

The conditions (a) and (b) lead then to the following equations (c) and (d)

$$\begin{aligned} &\text{c)}\ k[\sin\bar{\omega}\sin\bar{\omega}n_0-(\cos\bar{\omega}-1)\cos\bar{\omega}n_0] = (\cos\bar{\omega}-1)^2+\sin^2\bar{\omega}, \\ &\text{d)}\ \sin\bar{\omega}\cos\bar{\omega}n_0+(\cos\bar{\omega}-1)-\sin\bar{\omega}n_0 = 0. \end{aligned} \tag{146}$$

After considerable elementary trigonometric transformations, we shall arrive at a simpler equation — namely

$$\text{c}')\ k \sin\left(\bar{\omega} n_0 + \frac{\bar{\omega}}{2}\right) = 2 \sin \frac{\bar{\omega}}{2}, \qquad\qquad (147)$$

$$\text{d}')\ \cos\left(\bar{\omega} n_0 + \frac{\bar{\omega}}{2}\right) = 0.$$

The above equations will be solved with respect to the unknowns $\bar{\omega}$ and k. From (d′) it follows that

$$\bar{\omega} n_0 + \frac{\bar{\omega}}{2} = \frac{\pi}{2} + \nu\pi,$$

where ν is any integral number.
Hence

$$\bar{\omega} = \bar{\omega}_\nu = \frac{1 + 2\nu}{1 + 2n_0}\pi.$$

Substituting $\bar{\omega}$ into equation (c′), we shall then find the unknown k

$$k = k_\nu = \frac{2 \sin \frac{\bar{\omega}}{2}}{\sin\left(\bar{\omega} n_0 + \frac{\bar{\omega}}{2}\right)} = (-1)^\nu 2 \sin \frac{1 + 2\nu}{1 + 2n_0} \frac{\pi}{2}. \qquad (148)$$

It can easily be shown that in this case also the smallest non-negative value of k_ν will be obtained for the fundamental component of the control process — that is, for $\nu = 0$. The greatest admissible inclination of the time characteristic $x_2(n, 0)$ of an open control system is then k_0

$$k_0 = 2 \sin \frac{1}{1 + 2n_0} \cdot \frac{\pi}{2}. \qquad (149)$$

The function $k_0 = f(n_0)$ is shown in Fig. 11. From the plot of this function, we conclude that the greatest admissible inclination of the time characteristic of an open sampled-data control system decreases with increase in n_0 — that is, with increase of the delay introduced by the system.

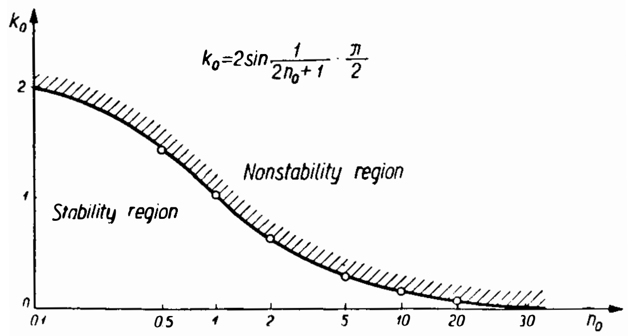


Fig. 11. Stability limit of an astatic sampled-data system

The method presented for determining stability is convenient primarily when the time characteristic $x_2(n, 0)$ of an open control system is determined experimentally. If, for this or that reasons, experimental determination of the above characteristic is impossible, the characteristic can be calculated analytically — for example, by means of the method presented in [15]. Then we use the equation of a sampled-data system

$$\{x_2(n, 0)\} = K_I(p, 0)\{x_1(n)\}$$

and we assume that

$$\{x_1(n)\} = \{1\} = \frac{p}{p-1}.$$

The characteristic $x_2(n, 0)$ of an open sampled-data control system may also be determined by the graphical-analytical method; namely, we

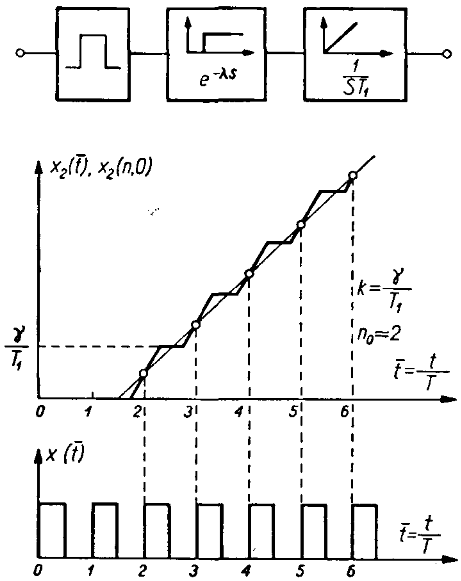


Fig. 12. Time characteristic of an astatic sampled-data system with a delaying element

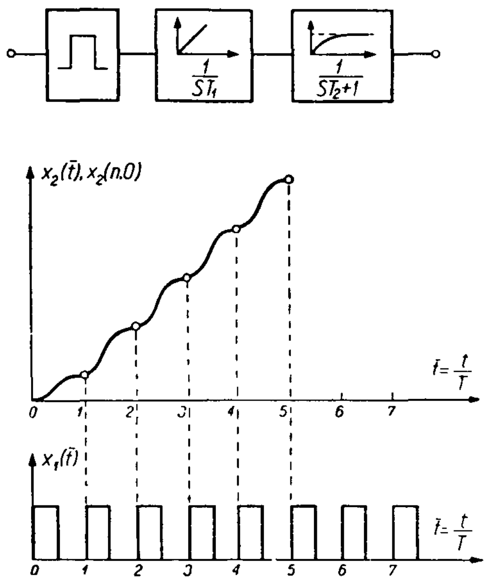


Fig. 13. Time characteristic of an astatic sampled-data system with an inertial element

may add graphically the responses $h_\gamma(t) = h(t) +- h(t - \gamma T)$ of the linear part of the system to rectangular pulses with the width γT which are displaced with respect one to another by the period T.

It is clear that an influence on the inclination of the characteristic $x_2(n, 0)$ is exerted by changing the width of the rectangular pulses — that is, by the change of the parametr γ. It can easily be seen that the inclination of $x_2(n, 0)$ decreases with a decrease in the parametr γ.

Simple astatic systems and their time characteristics $x_2(t)$, $x_2(n, 0)$ are pressented in Figs. 12 and 13.

Note that in the case of the system from Fig. 12, between the inclination k_0 of the characteristic $x_2(n, 0)$ and the width of the pulses, a simple relation holds

$$k_0 = \frac{\gamma}{T_1}.$$

Using this formula, we can find the greatest admissible width of pulses for a system with the characteristic shown in Fig. 12. The control system of gas pressure may serve as an example for a system with such a characteristic. An example for a system with the characteristic shown in Fig. 13 may be the temperature control system.

PART III

CALCULATION OF TRANSIENTS BY A NUMERICAL METHOD

1. INTRODUCTION

In dealing with the problems of transients, we often use approximate methods — for example, graphical-analytical methods which in several cases reduce and simplify the calculations. These methods are at present being developed by numerous authors and have found wide applications in practice. In particular, they are extensively applied in control engineering — namely, in the problems of control systems. Of interest here are the methods advanced by a Russian author, L. A. Bashkirov [2], which is designed for an approximate solution of differential equations with constant or variable coefficients, the method of U.A. Bailey and what is called the „method of moments".

An interesting approximate method for calculating transients was presented by Tustin [13]. This is a numerical method in which continuous functions are approximated by means of functions "constructed" of elementary functions in the form of triangles. Using such an approximation, Tustin calculates numerically the time waveforms in control systems.

Tustin's method could (as regards its mathematical aspect) be set into correspondence with the numerical calculation of the inverse Laplace transformation

$$f(t) = L^{-1}\{F(s)\},$$

where $F(s)$ is a rational function of the complex variable s, which characterizes the properties of the system under investigation which are of interest to us.

We shall show in Chapter 4 that this problem can be reduced to a numerical calculation of the integral equation of the form

$$\int_0^t f(\tau) f_2(t-\tau)\, d\tau = f_1(t),$$

where $f_1(t)$ and $f_2(t)$ are the given polynomials of the real variable t (Eqs. 162).

Approximating the continuous functions $f_1(t)$ and $f_2(t)$ by step functions, we can replace the above integral equation by a set of linear equations of the form

$$\sum_{m=0}^{n} f(m) f_2(n-m) = f_1(n), \qquad n = 0, 1, 2, \ldots, \tag{150}$$

from which the values of the function $f(n)$ can be found algebraically.

It should be noted that the expression (150) uniquely determines the product of the numerical operators $f(n)$ and $f_2(n)$ in the case in which $f(n)$, $f_2(n) = 0$ for $n < 0$. Hence the method presented above (which is a numerical method for calculating the convolution of functions) may be considered as one of the applications of numerical operators.

In the chapters which follow, we present a numerical method for calculating transients, which is based on the substitution of the expression (150) for the convolution of functions.

The theoretical foundations of the method and examples of applications are given in the book; however, no discussion on the accuracy of results is presented. An unquestionable advantage of the method is its great simplicity and the possibility of arriving easily at the results. On the other hand, it should be stressed that the method has a drawback in that it does not enable discussion as to the influence of the parameters of the system under investigation on the result obtained. Even so, in certain practical problems the method can be of great value.

2. APPROXIMATION OF A CONVOLUTION OF FUNCTIONS BY A QUOTIENT OF NUMERICAL OPERATORS

Assume that on the input of an arbitrary electric four-pole having linear elements a voltage of any shape is applied. This voltage may be approximated by a sequence of elementary rectangular pulses applied to the four-pole at equal time intervals $\Delta\tau$. The rectangular pulses cause certain elementary transients on the output of the four-pole. The input voltage of the four-pole may be considered as a result of the superposition of these elementary transients (Fig. 14).

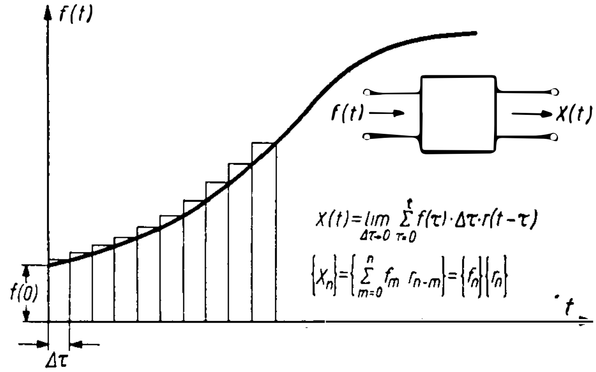


Fig. 14. Approximation of the time characteristic of a four-pole by means of a step function

Each rectangular voltage pulse having time of duration $\Delta\tau$ can be expressed as the difference of two unit-step jumps of voltage connected at the moments t and $t + \Delta\tau$.

If the reaction of the system to the unit-step jump is denoted by the symbol $h(t)$, then the elementary output signal $r(t)$ due to the application of a signal in the form of the rectangular pulse on the input of the four--pole will be expressed by the formula

$$r(t) = h(t) - h(t - \Delta\tau). \tag{151}$$

Assuming in turn that $\Delta\tau = 1$ and $t = n + \varepsilon$, we shall obtain

$$r(n + \varepsilon) = h(n + \varepsilon) - h(n + \varepsilon - 1), \tag{152}$$

where $n = 0, 1, 3, \ldots, 0 \leqslant \varepsilon < 1$.

The relation (152) may be written in terms of the numerical operator

$$\{r[n, \varepsilon]\} = \{h[n, \varepsilon]\} - \{h[n - 1, \varepsilon]\} = \}\nabla h[n, \varepsilon]\}. \tag{153}$$

If we confine ourselves to determining only certain particular points of the curve $r(t)$, then we can assume in Formula (153) that $\varepsilon = 0$, and consequently we shall obtain

$$\{r_n\} = \{h_n\} - \{h_{n-1}\} = \{\nabla h_n\}. \tag{154}$$

The output signal will — in accordance with the previous considerations — be a superposition of all the elementary signals $r(t)$ caused by the particular rectangular pulses. This signal may, then, be expressed by the formula

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{\tau=0}^{t} f(\tau)\,\Delta\tau\, r(t-\tau), \tag{155}$$

whence

$$x(t) = \int_0^t f(\tau)\, r(t-\tau)\, d\tau. \tag{156}$$

Formula (156) in the operator calculus is referred to as the "convolution of functions".

If, in Formula (155), we do not pass to the limit, we shall obtain an approximate value of the convolution — the higher the accuracy, the lower the width of rectangular pulses by which we approximate the function $f(t)$. The approximate value of the convolution may be represented by the formula

$$\{X[n,\varepsilon]\} = \left\{\sum_{m=0}^{n} f[m,\varepsilon]\cdot r[n-m,\varepsilon]\right\} \tag{157}$$

or, if we want to determine only certain particular points of the curve $X(t)$, we may assume that $\varepsilon = 0$ and accordingly Formula (157) will take the form

$$\{X_n\} = \left\{\sum_{m=0}^{n} f_m \cdot r_{n-m}\right\} \tag{158}$$

that is,

$$\{X_n\} = \{f_n\}\{r_n\} \tag{159}$$

or, after using Eq. (154)

$$\{X_n\} = \{f_n\}\{\nabla h_n\}. \tag{160}$$

Thus if we know the shape of the input voltage signal of the four-pole — the function $f(t)$, and if we know the reaction of the four-pole to the voltage jump — the function $h(t)$, then the signal at the output of the four-pole can be represented approximately by means of the product of the numerical operators $\{f_n\}$ and $\{\nabla h_n\}$ formed from the functions $f(t)$ and $h(t)$. Similarly, if we have an electric two-pole, then knowing the shape of the voltage signal of the source the function $f(t)$, and knowing the reaction of the two-pole to the voltage jump (the function $h(t)$), we can approximately determine the function of the current in the two-pole as the product of the numerical operators $\{f_n\}$ and $\{\nabla h_n\}$, which are formed from the functions $f(t)$ and $h(t)$.

The multiplication of numerical operators can easily be performed by means of the notation indicated in Table 1.

Table 1

Multiplication of numerical operators

$\hat{a}$	0	18	15	12	10	8	7	5
$\hat{b}$	0	1	2	3	4	5	0	1
	0	0	0	0	0	0	0	0
		0	18	15	12	10	8	7
			0	36	30	24	20	16
				0	54	45	36	30
					0	72	60	48
						0	90	75
							0	0
								0
$\hat{a}\,\hat{b}$	0	0	18	51	96	151	214	176

3. EXAMPLES

Example 1

A saw-like voltage was applied to a two-pole consisting of the elements R, L in a series connection (Fig. 15). Determine the function of the current $i(t)$.

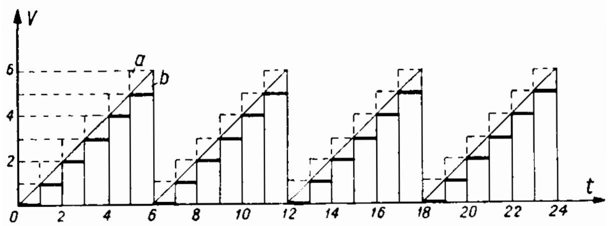


Fig. 15. Approximation of a triangular plot by means of a step function with deficiency and with excess

Solution

(a) We determine the reaction of the system to the unit-step function by well-known methods, obtaining

$$h(t) = i = \frac{1}{R}(1 - e^{-\frac{R}{L}t}) = \frac{1}{R}(1 - e^{-\frac{t}{T}}),$$

where $T = \dfrac{L}{R}$

(b) Knowing the numerical values of the constants R and T, we draw the function $h(t)$ on a profile paper (Fig. 16).

(c) We draw the plot of the driving voltage also on a profile paper (Fig. 15).

(d) We divide the time interval from the moment $t = 0$ to the steady-state moment of the plot $h(t)$ into a certain number of equal parts (the division on our drawing is into twenty five parts); we also divide the abscissa axis of the plot $h(t)$ into a certain number of equal parts (on our drawing we have one hundred such parts).

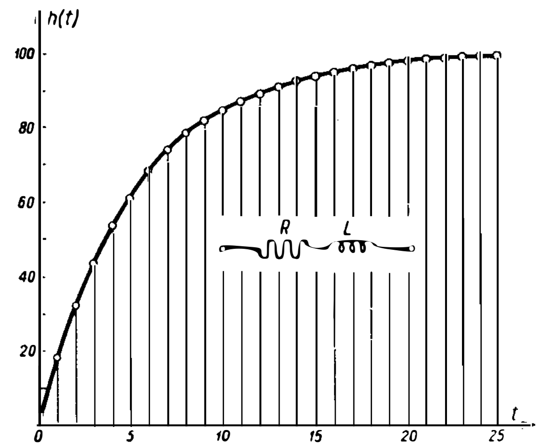


Fig. 16. Time characteristic of an RL two-pole

(e) The time axis of the plot $f(t)$ is divided in a manner identical with that for the time axis of the plot $h(t)$. The abscissa axis of the plot $f(t)$ may also be divided only into equal segments, but on an arbitrarily chosen scale.

(f) Form the value of the function $h(t)$, we form a numerical operator — in accordance with the division performed; we obtain

$$\{h_n\} = 0, 18, 33, 45, 55, 63, 70, 76, 81, 85, 87, 89, 91, 93, 95, 96, 97, 98, 99, 99, 99\ldots$$

(g) We determine the difference operator $\{\nabla h_n\}$:

$$\{\nabla h_n\} = 0, 18, 15, 12, 10, 8, 7, 6, 5, 4, 2, 2, 2, 2, 2, 1, 1, 1, 1, 0, 0, \ldots$$

(h) From the values of the function $f(t)$, we form a numerical operator

(a) with deficiency

$$\vec{f}_1 = \{f_{1n}\} = 0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5\ldots.$$

(b) with excess

$$\hat{f}_2 = \{f_{2n}\} = 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, \ldots$$

1, 2, 3, 4, 5, 6, ...

(i) By the "arithmetical method" we perform the multiplication of the operators

$$\text{a)}\ \hat{X}_1 = \{\nabla h_n\}\{f_{1n}\},$$

$$\text{b)}\ \hat{X}_2 = \{\nabla h_n\}\{f_{2n}\}.$$

The operators $\hat{X}_1$, $\hat{X}_2$ determine the current signal $i(t)$ sought for, with deficiency and with excess, respectively.

Note that we may avoid the determination of the signal with excess and with deficiency — that is, the twofold multiplication of the operators — if we form the intermediate operator $\{f_{int}\} = \left\{\frac{f_{1n} + f_{2n}}{2}\right\}$ and perform the multiplication $\{f_{int}\}$ $\{\nabla h_n\}$. In our example the intermediate operator f_{int} will take the form

$$\hat{f}_{int} = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2}, \ldots$$

Fig. 17 shows the diagrams of the current signal sought for; the curve (a) indicates the signal calculated with excess, the curve (b) — with deficiency, and the curve (c) — the intermediate signal. The results

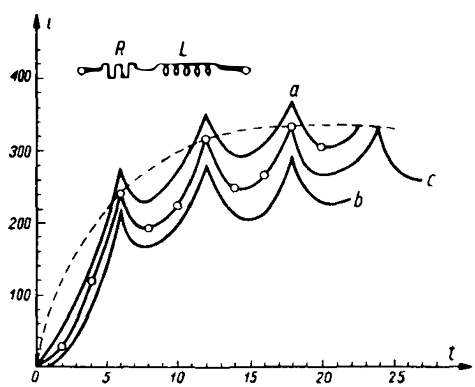


Fig. 17. Plot of the current in an RL two-pole caused by a saw-like voltage. The curve (a) shows the plot calculated with excess; the curve (b) — with deficiency; and the curve (c) — the mean plot. The circles indicate the values of the current as calculated analytically

obtained above were verified by means of the analytical method. The points of the plot $i(t)$ determined analytically are denoted on the Figure by circles.

Example 2

On the output of the four-pole R, C, which is shown in Fig. 18, a signal of the full-wave rectified voltage $U = |\sin \omega t|$ was applied. Determine the output function $U_2(t)$, with the assumption that the four-pole is open-circuited.

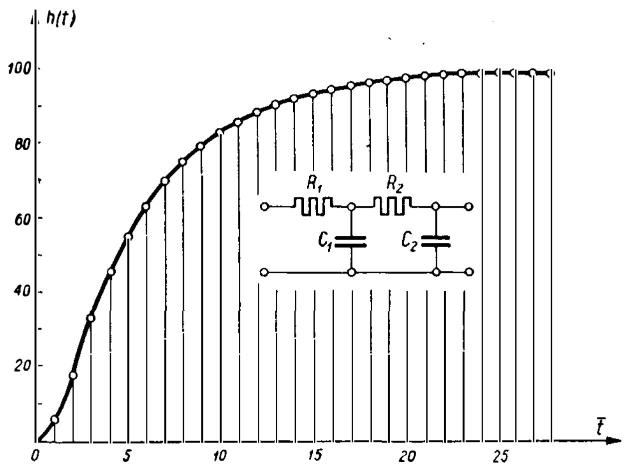


Fig. 18. Time characteristic of an RC two-section four-pole

Solution

The characteristic function of the four-pole is expressed by the formula

$$K(s) = \frac{1}{as^2 + bs + 1} = \frac{1}{a(s - \alpha_1)(s - \alpha_2)},$$

where

$$s \mathrel{\hat{=}} \frac{d}{dt} \mathrel{\hat{=}} j\bar{\omega}; \qquad a = R_1 C_1 R_2 C_2; \qquad b = R_1 C_1 + R_2 C_2 + R_1 C_2;$$

$$\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2 - 4a}}{2a}.$$

Hence the response of the four-pole to the excitation by a unit-step function will be

$$h(t) = \frac{1}{a}\left[\frac{1}{\alpha_1 \alpha_2} + \frac{e^{\alpha_1 t}}{\alpha_1(\alpha_1 - \alpha_2)} + \frac{e^{\alpha_2 t}}{\alpha_2(\alpha_2 - \alpha_1)}\right].$$

The function $h(t)$ with the values assumed for R_1, R_2, C_1, C_2 is given in Fig. 19.

In accordance with the chosen division of the abscissa axes and the coordinate axes, we obtain the following numerical sequence formed from the values of the function $h(t)$

$$\{h_n\} = 0, 6, 18, 34, 46, 54, 62, 69, 75, 80, 83, 86, 89, 91, 92, 95, 96, 97, 98, 99, \ldots$$

and after performing the difference operation

$$\{\nabla h_n\} = 0, 6, 12, 16, 8, 8, 7, 6, 5, 3, 3, 2, 1.5, 1.5, 1, 1, 1, 1, 0.5, 0.5, 0, 0, \ldots$$

The voltage $U = f(t)$ is replaced by the following sequence

$$\{f_{n\,int}\} = \left\{\frac{f_{1n} + f_{2n}}{2}\right\} = 26, 71, 97, 97, 71, 26, 71, 97, 97, 71, 26, \ldots$$

Thus, after performing the multiplication,

$$\{U_2(n)\} = \{\nabla h_n\}\{f_{n\,int}\} = 0, 156, 738, 1850, 3194, 4202, 4290, 4211, 4204, 4741$$
$$5664, 6265, 6175, 5652, 5317, 5662, 6327, 6873, 6673,$$
$$6088, 5705, 6002, 6707, 7067, \ldots$$

The voltage signal $U_2(t)$ is shown in Fig. 19.

Example 3

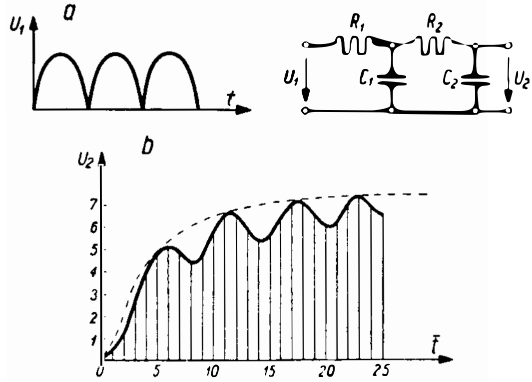


Fig. 19. Plot of the voltage on the output of an RC two-section four-pole caused by a full-wave rectified sinusoidal voltage

A sequence of rectangular pulses was applied on the input of the RC amplifier presented in Fig. 20. Determine, by the numerical method, the output function for two cases — namely, for a one-stage amplifier and a two-stage amplifier.

Solution

The characteristic function of a one-stage amplifier with the assumption of a small input capacity of the valve, will take the form

$$K_1(s) = \frac{sk_0}{s + \frac{1}{T}},$$

where

$$k_0 = S_i R_0, \quad T = C_2 (R_s + R_{ai});$$

$$R_{ai} = \frac{R_a \varrho_a}{R_a + \varrho_a}; \quad R_0 = \frac{R_{ai} R_s}{R_{ai} + R_s};$$

S_i is the inclination of the characteristic.

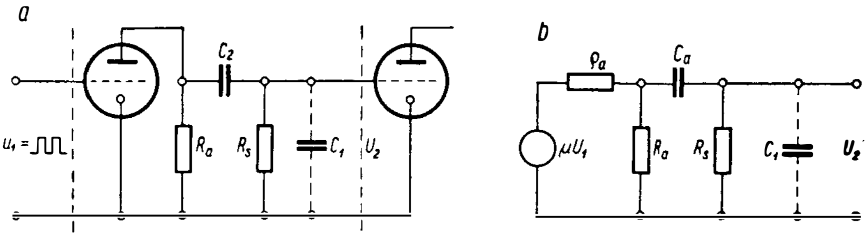


Fig. 20. Time characteristic of a one-stage and a two-stage resistance amplifier

The response of the system of a one-stage amplifier to a unit-step function may easily be determined in terms of the Laplace transformation

$$h_1(t) = L^{-1}\left\{\frac{K(s)}{s}\right\} = k_0 e^{-\frac{t}{T_2}}.$$

In the case of a two-stage amplifier the characteristic function $K(s)$ will be equal to the product

$$K(s) = K_1(s) \cdot K_2(s),$$

and if the stages are identical

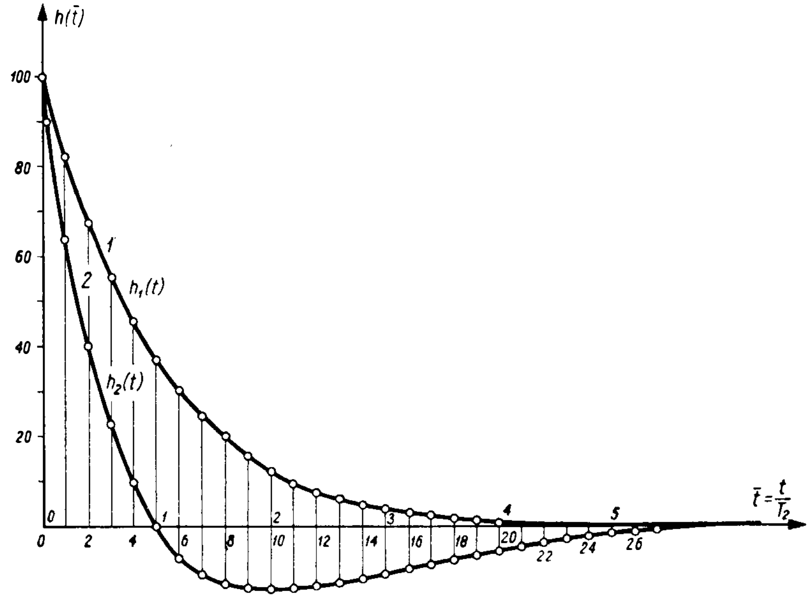
$$K(s) = K_1^2(s) = K_0^2 \frac{s^2}{\left(s + \frac{1}{T}\right)^2}.$$

Hence

$$h_2(t) = L^{-1}\left\{k_0^2 \frac{s^2}{\left(s + \frac{1}{T}\right)^2}\right\} = k_0^2 (1 - \alpha) e^{-\alpha t},$$

where $a = \frac{1}{T}$

Figure 21 shows the function $h_1(t)$ and $h_2(t)$ with tne assumption that $R = 100$ kΩ, $R_a = 30$ kΩ, $R_i = 30$ kΩ, $C_2 = 30000$pF.



From the functions $h_l(t)$ and $h_2(t)$, in accordance with the division of the time axis and abscissa axis as assumed, we form the following two numerical sequences

$$\{h_{1n}\} = 100, 83, 67, 55, 45, 37, 30, 25, 10, 16, 13, 20, 8, 6, 5, 4, 3, 2, 1, 0, 0, \ldots,$$

$$\{h_{2n}\} = 100, 65, 40, 24, 10, 0, -6, -10, -12, -12.5, -13, -11, -10, -9, \ldots.$$

Further, we determine $\{\nabla h_{1n}\}$ and$\{\nabla h_{2n}\}$;

$$\{\nabla h_{1n}\} = 100, -17, -16, -12, -10, -8, -7, -5, -5, -4, -3, -2, -2, -1, -1, -1, -1, 0, 0, \ldots,$$

$$\{\nabla h_{2n}\} = 100, -35, -25, -16, -14, -10, -6, -4, -2, -0.5, -0.5, 0.5, 50, 1, 1, \ldots.$$

The voltage $h_1(t)$ is replaced by the following numerical sequence

$$\{f_n\} = 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots.$$

Then, in order to determine the functions $U_2(t)$ and $U_3(t)$ for a one-stage amplifier and a two-stage amplifier, we perform the multiplication

$$\{U_2(n)\} = \{\nabla h_{1n}\} \cdot \{f_n\}, \quad \text{and} \quad \{U_3(n)\} = \{\nabla h_{2n}\} \cdot \{f_n\}.$$

The results are shown in Figs. 22 and 23.

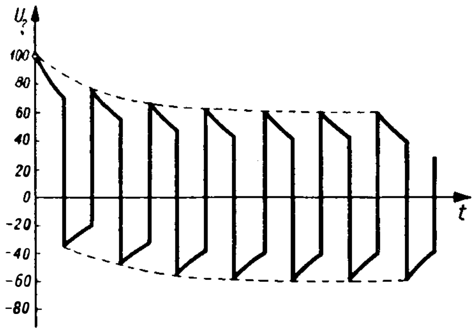


Fig. 22. Sequence of rectangular pulses after passing through a one-stage resistance amplifier

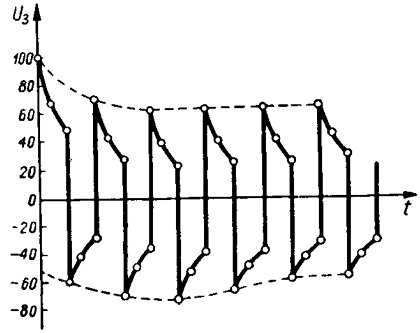


Fig. 23. Sequence of rectangular pulses after passing through a two-stage resistance amplifier

4. NUMERICAL CALCULATION OF THE RESPONSE OF A SYSTEM TO A UNIT-STEP EXCITATION

The numerical calculation of transients by the method presented in the preceding chapter requires knowledge of the excitation function and the time characteristic $h(t)$ of the electric system. It should be noted that the determination of the characteristic $h(t)$ in the case of systems containing a great number of elements (owing to which the degree of the denominator in the characteristic function $K(s)$ of the system is high) may present serious difficulties. Thus if great accuracy is not required, it is convenient in such cases to calculate the time characteristic $h(t)$ by the numerical method also. Since the function $h(t)$ is related with the characteristic function $K(s)$ of the system by the Laplace transformation

$$h(t) = L^{-1}\left\{\frac{K(s)}{s}\right\};$$

then the numerical determination of the function $h(t)$ may be subordinated to a wider problem — namely, to the numerical method of calculating the results of the inverse Laplace transformation. Below is presented the idea of the numerical method of calculating the Laplace transformation, which constitutes a continuation of the method described in the preceding chapter.

In the case in which $K(s)$ is a rational function, which occurs when the electric system under consideration consists of lumped elements, we have

$$\frac{K(s)}{s} = \frac{b_0 s^\nu + \ldots + b_\nu}{a_0 s^\mu + \ldots + a_\mu} = \frac{b_0 \frac{1}{s^{\mu-\nu+1}} + \ldots + b_\nu \frac{1}{s^{\mu+1}}}{a_0 \frac{1}{s} + \ldots + a_\mu \frac{1}{s^{\mu+1}}}.$$

Then $h(t)$ satisfies the equation

$$L\{h(t)\} = \frac{L\{f_1(t)\}}{L\{f_2(t)\}}, \tag{161}$$

where the functions $f_1(t)$ and $f_2(t)$ are the polynomials

$$\begin{aligned} f_1(t) &= \frac{b_0}{(\mu-\nu)!} t^{\mu-\nu} + \frac{b_1}{(\mu-\nu+1)!} t^{\mu-\nu+1} + \ldots + \frac{b_\nu}{\mu!} t^\mu, \\ f_2(t) &= a_0 + \frac{a_1}{1!} t + \ldots + \frac{a_\mu}{\mu!} t^\mu. \end{aligned} \tag{162}$$

By virtue of Eq. (161), the function $h(t)$ being sought will evidently satisfy the integral formula

$$\int_0^t h(\tau) \cdot f_2(t-\tau)\, d\tau = f_1(t),$$

which in accordance with the considerations from the preceding chapter can be calculated in an approximate manner by the numerical operator method. The problem reduces to determining the quotient of two numerical operators

$$\{h(\tau n)\} = \frac{\{f_1(\tau n)\}}{\{\nabla \varphi(\tau n)\}},$$

where $\nabla\varphi(\tau n) = \varphi(\tau n) - \varphi[\tau(n-1)]$ is a sequence formed from the function $\varphi(t) = \int_0^t f_2(\xi) d\xi$, and $f_1(\tau n)$ is a sequence of the values of the function $f_2(t)$.

The method presented above was checked by the author by way of several practical examples. As a result of the investigations effected, it has been proved that the method is convenient in applications when the function $h(t)$ sought for is what is called a "slowly varying" function, and this takes place in $R\,C$ or $R\,L$ systems [3]. To obtain sufficiently accurate results, the method presented requires, however, that the calculations be performed with a degree of accuracy up to the fourth or even

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[3]) The influence of the accuracy of the terms-sequences $\{a_n\}$ and $\{b_n\}$ on the accuracy of the terms of the quotient $\{x_n\} = \frac{\{a_n\}}{\{b_n\}}$ may in relatively simple manner be determined from Formulae (9) with the assumption that $p = k = 0$.

fifth decimal point, and consequently the calculations cannot be performed by means of a sliding scale. The method is convenient for calculating the operation inverse to convolution in the cases in which the functions $f_1(t)$ and $f_2(t)$ are given in the form of diagrams.

It is also possible, as regards the calculation of the Laplace transformation, to use what is called the "method of moments" [16]. This method is set out below.

Let the symbol $\delta(t)$ be the pulse Dirac delta function. This function is usually determined in the engineering literature as the limit of a certain continuous function $\delta(t, \lambda)$, for instance

$$\delta(t) = \lim_{\lambda\to\infty} \delta(t, \lambda) = \lim_{\lambda\to\infty} \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 t^2}.$$

It is known that the arbitrary function $x(t)$ of the real variable t occurring in physics problems can be expressed by means of the pulse function $\delta(t)$ as

$$x(t) = \frac{\int\limits_0^\infty x(\tau)\,\delta(t-\tau)\,d\tau}{\int\limits_0^\infty \delta(t-\tau)\,d\tau}.$$

If the pulse function $\delta(t)$ in the above formula is replaced by the continuous function $g(t)$ possessing a "sharp" maximum at the point $t = \tau$, then we shall obtain an approximate formula. As $g(t)$ we shall assume the function $t^n e^{-at}$

$$g(t) = t^n e^{-\alpha t}.$$

This function has maximal value at the point $t = n/\alpha$. We shall then obtain the relation

$$x(^n/\alpha) \approx \frac{\int\limits_0^\infty x(\tau)\cdot \tau^n e^{-\alpha\tau}\,d\tau}{\int\limits_0^\infty \tau^n e^{-\alpha\tau}\,d\tau} = \frac{x_{\alpha n}}{n!/\alpha^{n+1}},$$

where

$$x_{\alpha n} = \int\limits_0^\infty x(\tau)\cdot \tau^n e^{-\alpha\tau}\,d\tau.$$

It appears that, because of the asymmetry of the function $g(t)$ near the maximal value, a more accurate formula will be obtained if, in $x(n/a)$, we assume $n + 1$ instead of n

$$x\left(\frac{n+1}{\alpha}\right) = \frac{x_{\alpha n}}{n!/\alpha^{n+1}}. \tag{163}$$

Let us observe that the expression x_{an} is related in a simple manner with the Laplace transformation of the function $x(t)$. For we have

$$\left[\frac{d^n X(s)}{ds^n}\right]_{s=\alpha} = \frac{d^n}{d\alpha^n}\int_0^\infty x(t)\, e^{-\alpha t}\, dt = \int_0^\infty x(t)\cdot(-1)^n t^n e^{-\alpha t}\, dt = (-1)^n x_{\alpha n}, \quad (164)$$

where $X(s)$ is the Laplace transformation of the function $x(t)$.

If for α we assume the real number $\frac{1}{\tau}$

$$\alpha = \frac{1}{\tau},$$

then taking into consideration Equations (164) and (163), we shall obtain

$$x[(n+1)\tau] \approx (-1)^n \frac{1}{n!\,\tau^{n+1}}\left[\frac{d^n X(s)}{ds^n}\right]_{s=\frac{1}{\tau}}. \quad (165)$$

This formula determines the relation between the values of the function $x(t)$ at the points τ, $2\tau, \ldots (n+1)\tau$, and its Laplace image $X(s)$.

Since $\left[\frac{d^n X(s)}{ds^n}\right]_{s=\frac{1}{\tau}}$ is a coefficient of the n-th power in the expansion $X(s + {}^1/\tau)$

$$X(s + 1/\tau) = c_0 + c_1 s + c_2 s^2 \ldots + c_n s^n + \ldots,$$

thus Formula (165) may also be written as

$$x[(n+1)\tau] \approx (-1)^n \frac{c_n}{\tau^{n+1}}. \quad (166)$$

Using Formula (166) we may, in a simple manner, determine the sequence of approximated values of the function $x(t)$ directly from knowledge of the coefficients of the expansion of $X(s + {}^1/\tau)$ into a power series. This procedure will be explained by the simplest possible example. Let us consider the function

$$X(s) = \frac{1}{s-1}.$$

Assuming that $\tau = 10^{-1}$ sec, we calculate

$$X(s + 1/\tau) = X(s + 10) = \frac{1}{s+9} = \frac{1}{9} - \frac{s}{9^2} + \frac{s^2}{9^3}\ldots$$

We thus have

$$c_n = (-1)^n \frac{1}{9^{n+1}}:$$

The final result is obtained by the use of the Formula (107)

$$x(0.1) \approx \frac{10}{9} \approx 1.11,$$

$$x(0,2) \approx \frac{100}{81} \approx 1.235,$$

$$x(0.3) \approx \frac{1000}{730} \approx 1.371,$$

$$x(0.4) \approx \frac{10000}{6560} \approx 1.525,$$

. . . .

In order to check the accuracy of the result obtained, we now calculate the values of $x[(n + 1)\tau]$ analytically.

Since $x(t) = L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$,

$$x(0.1) = e^{0,1} \approx 1.105,$$
$$x(0.2) = e^{0,2} \approx 1.222,$$
$$x(0.3) = e^{0,3} \approx 1.350,$$
$$x(0,4) = e^{0,4} \approx 1.492,$$

. . . .

Comparing the results obtained analytically with those yielded by the approximate method, we can state that the maximal error of the approximate calculations does not exceed 2.5%.

In practical problems the function $X(s)$ most frequently has the rational form

$$X(s) = \frac{P(s)}{Q(s)} = \frac{\bar{b}_0 + \bar{b}_1 s + \ldots + \bar{b}_\nu s^\nu}{\bar{a}_0 + \bar{a}_1 s + \ldots + \bar{a}_\mu s^\mu}; \quad \mu \geqslant \nu.$$

In order to calculate the displaced function $X(s + 1/\tau)$, we have to determine the expansion of a certain number of expressions $\left(s + \frac{1}{\tau}\right)^m$. We use in these calculations the Newton formula

$$(a + b)^m = a^m + ma^{m-1} b + \frac{m(m-1)}{2!} a^{m-2} b^2 + \ldots + ma \cdot b^{m-1} + b^m.$$

If the function $X(s)$ is rational, then $X\left(s + \frac{1}{\tau}\right)$ is evidently a rational function, also

$$X(s + 1/\tau) = \frac{b_0 + b_1 s + \ldots + bs^\nu}{a_0 + a_1 s + \ldots + as^\mu}; \quad a_0 \neq 0; \quad \mu \geqslant \nu.$$

The coefficients c_n of the expansion of the rational function $X(s + {}^1/\tau)$ into a power series can be simply calculeted by "dividing" (in the sense of numerical operators) the sequence $\{b_n\} = b_0,\ b_1, \dots b_\nu,\ 0,\ 0$ by the sequence $\{a_n\} = a_0, a_1, \dots a, 0, 0$, since we have

$$\{c_n\} = \frac{\{b_n\}}{\{a_n\}} = \frac{b_0}{a_0},\quad \frac{a_0 b_1 - b_0 a_1}{a_0^2},\ \dots \tag{167}$$

The calculation of the quotient (167) can conveniently be performed according to the scheme shown in Table 2.

$$\begin{array}{rrrrl}
18; & 51; & 96; & 151; \dots & \big|\ \underline{1;\ 2;\ 3;\ 4;\ \dots} \\
- & & & & \\
\underline{18;} & \underline{36;} & \underline{54;} & \underline{72; \dots} & \big|= 18; 15; 12; 10; \dots \\
= & 15; & 42; & 79; \dots & \\
- & & & & \\
 & \underline{15;} & \underline{30;} & \underline{45; \dots} & \\
 & = & 12; & 34; \dots & \\
 & - & & & \\
 & & \underline{12;} & \underline{24; \dots} & \\
 & & = & 10 & \\
 & & - & & \\
 & & & \underline{10} & \\
 & & & = &
\end{array}$$

Concluding the description of the method presented above, it is worth while to discuss the problem of selection in the calculations of the number τ. Since τ is the distance between the calculated values of $x[(n + 1)\tau]$ of the sought for function $x(t) = L^{-1}\{X(s)\}$, then it is evident that the greater the accuracy of the result obtained, the smaller the number τ selected. It should be noted that an erroneous τ may lead to erroneous results, especially when the function $x(t)$ is a "rapidly varying" function. Therefore, before starting the calculations, it is necessary if we have no sufficient information as to the plot of the function $x(t)$, and in order to ensure the correctness of the result, to carry out calculations corresponding to two different values of the number τ (for example, $\tau = {}^1/_{10}$ sec and $\tau = {}^1/_{20}$ sec). If however, the calculation of the transformation $L^{-1}\{X(s)\}$ is effected in connection with the solution of a concrete physical problem (for example, in connection with the determination of the response of a system to a unitstep excitation), we have from the conditions of the given problem a sufficient orientation as to the choice of the number τ.

Finally, it should be noted that — in accordance with the quotient (167) — the method presented here also leads to an operation analogous to the division of numerical operators. The application of the method to determining the time characteristic is illustrated by the following example.

Example

Determine by the approximate method the response $h(t)$ of the system to a unit-step excitation, with the assumption that the characteristic function of the system $K(s)$ is expressed by

$$K(s) = \frac{10 + 2s + 7s^2}{24 + 26s + 9s^2 + s^3}.$$

Solution

It is known that the response of a system to a unit-step excitation is related with the characteristic function $K(s)$ of the system by the Formula

$$h(t) = L^{-1}\left\{\frac{K(s)}{s}\right\}.$$

Thus, in order to determine the points of the plot $h(t)$, we investigate the expression

$$X(s) = \frac{K(s)}{s} = \frac{10 + 2s + 7s^2}{24s + 26s^2 + 9s^3 + s^4}.$$

Assuming that $\tau = 0.1$ sec, we calculate the displaced function $X(s + 10)$; after computations, we obtain

$$X(s+10) = \frac{730 + 142s + 7s^2}{21840 + 7244s + 896s^2 + 49s^3 + s^4}.$$

Then, dividing the sequences of coefficients

$$\{b_n\} = 730; 142; 7; 0; 0; 0; \ldots,$$

$$\{a_n\} = 21840; 7244; 896; 49; 1; 0; 0; 0; \ldots,$$

we obtain

$$\{c_n\} = \frac{\{b_n\}}{\{a_n\}} = 0.334 \cdot 10^{-1}; 0.458 \cdot 10^{-2}; 0.485 \cdot 10^{-3}; 1.00 \cdot 10^{-4}; 1.64 \cdot 10^{-5} \ldots$$

By virtue of Formula (166), we then have

$$h(0.1) \approx \frac{0,334 \cdot 10^{-1}}{\tau} = 0.334,$$

$$h(0.2) = \frac{0,458 \cdot 10^{-2}}{\tau^2} = 0.458,$$

$$h(0.3) \approx \frac{0,485 \cdot 10^{-3}}{\tau^3} = 0.485,$$

$$h(0,4) \approx \frac{1.00 \cdot 10^{-4}}{\tau^4} = 1.00,$$

$$h(0,5) \approx \frac{1.64 \cdot 10^{-5}}{\tau^5} = 1.64,$$

. . . .

Table of numerical operators

No.	$\{a_n\}$ $n \geqslant 0$	$F(p)$
1	$\{1\}$	$\frac{p}{p-1}$
2	$\{n\}$	$\frac{p}{(p-1)^2}$
3	$\{n^2\}$	$\frac{p}{(p-1)^3}(p+1)$
4	$\{n^3\}$	$\frac{p}{(p-1)^4}(p^2+4p+1)$
5	$\{n^4\}$	$\frac{p}{(p-1)^3}(p^3+11p^2+11p+1)$
6	$\{n^{(2)}\} = \left\{\frac{n(n-1)}{2!}\right\}$	$\frac{p}{(p-1)^3}$
7	$\{n^{(3)}\} = \left\{\frac{n(n-1)(n-2)}{3!}\right\}$	$\frac{p}{(p-1)^4}$
8	$\{n^{(m)}\} = \left\{\frac{n(n-1)\ldots(n-m+1)}{m!}\right\}$	$\frac{p}{(p-1)^{m+1}}$
9	$\{C^{\alpha n}\}$	$\frac{p}{p-C^{\alpha}}$
10	$\{(-C)^{\alpha n}\}$	$\frac{p}{p+C^{\alpha}}$
11	$\{nC^{\alpha(n-1)}\}$; $n \geqslant 1$	$\frac{p}{(p-C^{\alpha})^2}$
12	$\{(n-1)C^{\alpha(n-2)}\}$; $n \geqslant 2$	$\frac{1}{(p-C^{\alpha})^3}$
13	$\{n^2 \cdot C^{\alpha(n-1)}\}$; $n \geqslant 1$	$\frac{p}{(p-C^{\alpha})^3}(p+C^{\alpha})$
14	$\{n^{(2)}C^{\alpha(n-2)}\}$; $n \geqslant 2$	$\frac{p}{(p-C^{\alpha})^3}$
15	$\{n^{(m)}C^{\alpha(n-m)}\}$; $n \geqslant m$	$\frac{p}{(p-C^{\alpha})^{m+1}}$
16	$\{e^{\alpha n}\}$	$\frac{p}{p-e^{\alpha}}$
17	$\{e^{jxn}\}$	$\frac{p}{p-e^{jx}}$

Table of numerical operators

No.	$\{a_n\}$ $\quad n \geqslant 0$	$F(p)$
18	$\{\sin xn\}$	$\frac{p \sin x}{p^2 - 2p \cos x + 1}$
19	$\{\cos xn\}$	$\frac{p(p - \cos x)}{p^2 - 2p \cos x + 1}$
20	$\{\cos \pi n\} = \{(-1)^n\}$	$\frac{p}{p+1}$
21	$\{C^{\alpha n} \cdot \sin xn\}$	$\frac{C^\alpha p \sin x}{p^2 - 2pe^\alpha \cos x + C^{2\alpha}}$
22	$\{C^{\alpha n} \cos xn\}$	$\frac{p(p - C^\alpha \cos x)}{p^2 - 2p\, C^\alpha \cos x + C^{2\alpha}}$
23	$\{\sinh xn\}$	$\frac{p \sinh x}{p^2 - 2p \cosh x + 1}$
24	$\{\cosh xn\}$	$\frac{p(p - \cosh x)}{p^2 - 2p \cosh x + 1}$
25	$\{C^{\alpha n} \cosh xn\}$	$\frac{C^\alpha p \sinh x}{p^2 - 2p\, C^\alpha \cosh x + C^{2\alpha}}$
26	$\{C^{\alpha n} \cosh xn\}$	$\frac{p(p - C^\alpha \cosh x)}{p^2 - 2p\, C^\alpha \cosh x + C^{2\alpha}}$
27	$\frac{1}{1 - C^\alpha}\{1 - C^{\alpha n}\}$	$\frac{p}{(p - C^\alpha)(p - 1)}$
28	$\frac{1}{1 - C^\alpha}\{1 - C^{\alpha(n-1)}\}\ ;\ n \geqslant 1$	$\frac{1}{(p - C^\alpha)(p - 1)}$
29	$\frac{1}{C^{\alpha_1} - C^\alpha}\{C^{\alpha_1} - C^{\alpha n}\}$	$\frac{p}{(p - C^\alpha)(p - C^{\alpha_1})}$
30	$\frac{\{n\}}{1 - C^\alpha} - \frac{\{1 - C^{\alpha n}\}}{(1 - C^\alpha)^2}$	$\frac{p}{(p - C^\alpha)(p - 1)^2}$
31	$\left\{\frac{1}{n!}\right\}$	$e^{\frac{1}{p}} = e^q$
32	$\left\{\frac{n+1}{n!}\right\}$	$(q + 1)\, e^q$
33	$\left\{\frac{\sin xn}{n!}\right\}$	$e^{q \cos x} \cdot \cos(q \sin x)$
34	$\left\{\frac{\cos xn}{n!}\right\}$	$e^{q \cos x} \cdot \sin(q \sin x)$
35	$\left\{\frac{\sin xn}{n}\right\}$	$\arctan \frac{\sin x}{p - \cos x}$

Carrying out the calculations with the assumption that $\tau = 0.05$ sec., or using the analytical method, we may confirm that the results obtained are sufficiently accurate. After computing a large number of points of the function $h(t)$, we can then draw its plot.

ANNOTATIONS

Deduction of the formula determining the transfer function of a feedback sampled-data system.

If the transfer function of an open sampled-data system at the point "a" (Fig. 6) is denoted by the symbol $K_i\,(p, \varepsilon)$, then we shall obtain the relation

$$U_{out}(p, \varepsilon) + \frac{1}{k_u} K_t(p, \varepsilon) \cdot U_1(p). \qquad \text{(a)}$$

This equation is of course also correct for $\varepsilon = 0$, thus we have

$$U_{out}(p, 0) = \frac{1}{k_u} K_l(p, 0)\, U_1(p). \qquad \text{(b)}$$

Assume that the excitation $U_0(n, \varepsilon)$, was applied at the output of the system. Then, if the feedback is negative, we shall obtain

$$U_{in}(n, \varepsilon) = U_0(n, \varepsilon) - U_{out}(n, \varepsilon),$$

thus, also

$$U_{in}(p, \varepsilon) = U_0(p, \varepsilon) - U_{out}(p, \varepsilon) \qquad \text{(c)}$$

and at the moment of the appearance of pulses

$$U_{in}(p, 0) = U_0(p, 0) - U_{out}(p, 0). \qquad \text{(d)}$$

Let us further observe that U_1 is related with U_{we} only at the moments $\varepsilon = 0$, therefore we have

$$U_1(p) = k_u\, U_{in}(p, 0). \qquad \text{(e)}$$

Taking now into consideration in (b) the relations (d) and (e), we shall obtain

$$U_{out}(p, 0) = K_l(p, 0)\, [U_0(p, 0) - U_{out}(p, 0)],$$

and hence

$$U_{out}(p, 0) = \frac{K_l(p, 0)}{1 + K_l(p, 0)}\, U_0(p, 0). \qquad \text{(f)}$$

Taking in turn into account the relation (b) we shall find

$$U_1(p) = \frac{k_u}{1 + K_l(p, 0)}\, U_0(p, 0),$$

hence after using (a) we shall finally obtain

$$U_{out}(p, \varepsilon) = \frac{K_i(p, \varepsilon)}{1 + K_I(p, 0)} U_0(p, 0). \tag{g}$$

Formula (g) will be called the equation of a feedback sampled-data system, and the expression

$$K_{s,i}(p, \varepsilon) = \frac{K_i(p, \varepsilon)}{1 + K_I(p, 0)} \tag{h}$$

will be given the name of the transfer function or the characteristic function of a feedback sampled-data system.

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THEORY AND DESIGN OF SAMPLED — DATA CONTROL SYSTEMS [1])

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This paper presents the foundations of the theory and the principles od designing sampled-data control systems. The paper consists of three parts; Part I is concerned with basic information concerning the mathematical method on which is based the analysis of sampled-data systems; Part II covers the principles of the theory of sampled-data systems containing no feedbacks; in Part III an outline is given of the theory and principles of designing sampled-data control systems.

Author

INTRODUCTION

We may distinguish three principal stages in the development of the theory of sampled-data control systems.

In the first stage, sampled-data control systems were investigated by the analytical-graphical method. That method consisted in analytical determination of elementary waveforms corresponding to single rectangular pulses, and then in the graphical summing up of such waveforms.

In the second stage, the classical method used for the investigation of sampled--data control systems, applying the theory of differential equations. This method was based on the statement that the control process in a sampled-data control system can be determined by means of linear difference equations with constant coefficients.

In the third and last stage of investigating sampled-data control systems, the operator method was applied, which makes use of the discrete or summation Laplace transformation [1], [2], [3], [4].

The last method, which was recently worked out and developed by a Russian author, Y. Z. Tsypkin, yielded the foundations for a broad development of the theory of sampled-data control systems. The main advantage of this method lies in the fact that it makes possible to consider a large class of sampled-data control systems on the hasis of the same theoretical principles. Another advantage is that this method unables us to use, in the theory of sampled-data systems, several notions (such as, for example, time characteristic, frequency characteristic, operator transfer function, spectrum transfer function, etc.) which are well-known from the theory of dynamical systems. In spite of such advantages, Tsypkin's method has a serious and basic disadvantage — namely, it is based on the summation Laplace transformation, which is a special

[1]) Rozprawy Elektrotechniczne (Vol. III, No. 4, 1957, pp. 459—530).

mathematical apparatus, not applied to other problems, and thus unknown to a large group of engineers.

In should also be noted that the summation Laplace transformation imposes an essential restrictions as regards the class of functions to be considered, and therefore limits the range of applications of the method.

For the reasons stated, to establish the theory of sampled-data control systems on the method of the summation Laplace transformation is, in the opinion of the present author, not justified from the methodological point of view.

The theory of sampled-data control systems presented in this paper is based on the method of the integral Laplace transformation which is in common use in the theory of dynamical systems. Moreover, care has been taken to make the theory of sampled-data control system as similar as possible to the theory of control systems with continuous action.

PART ONE

OPERATIONS ON STEP FUNCTIONS

1. THE NOTION OF A STEP FUNCTION

A step function will be called such a real function $f(t)$ of the real variable t, as is constant in each interval $[nT,(n+1)T]$, where n is an integral number, and T is a constant greater than zero (Fig. 1). The constant T is called the "parameter of the step function". The step function is usually denoted by the symbol $\underline{\ulcorner} f(t)$; however, this notation is not convenient in practice. In this paper we shall denote this function in a simpler manner, namely, by the symbol $f[t]$.

Let us assume that the values of the step function at the points of discontinuities are its right-side limits at these points — that is

$$f[nT] = \lim_{\varepsilon \to 0} f[nT + \varepsilon] = f[nT + 0]; \qquad \varepsilon \geqslant 0. \tag{1}$$

Notice that by means of a simple substitution

$$\bar{t} = \frac{t}{T} \tag{2}$$

the step function can be reduced to a function which is constant in unit intervals $[n, n+1]$, thus to a step function $f[\bar{t}]$ whose parameter is the number $T = 1$.

For simplicity, we shall confine ourselves in what follows to investigating step functions $\bar{f}[t]$ with a unit parameter. This restriction obviously entails no limitations as regards the generality of considerations, since after performing any operations on the functions $f[\bar{t}]$, we may always transpose the result — by substituting (2) — on the functions $f[\bar{t}]$ with an arbitrary parameter T. The step function $f[\bar{t}]$ can always be uniquely determined by means of the sequence $f(n)$ formed from its values. Notice that the four arithmetical operations performed on the functions $f[\bar{t}]$ correspond to (are isomorphic with) four analogous operations performed on the sequences $f(n)$. This property holds also for any difference operations (displacement, summing, difference operation). Owing to this property, the step function will be simply denoted by the symbol $f(n)$ instead of $f[\bar{t}]$.

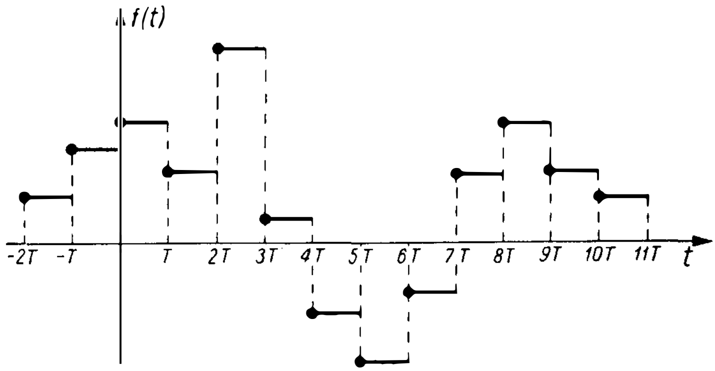


Fig. 1. Step function

2. OPERATIONS OF DISPLACEMENT, DIFFERENCE AND SUMMING

We define certain operations on the set of step functions $f[\bar{t}]$ — namely, displacement, difference and summing.

The displacement is an operation setting into correspondence the step function $f[\bar{t}]$ with the function $f[\bar{t} + k]$, where k is any intergral number. If, then, the displacement operation is denoted by the symbol $T^k f[\bar{t}]$, then we shall obtain the formula

$$T^k f[\bar{t}] = f[\bar{t} + k]. \tag{3}$$

The difference operation is that which sets into correspondence the function $f[\bar{t}]$ with the function in the form of the difference $f[\bar{t} + 1] - f[\bar{t}]$. This operation is usually denoted by the symbol $\Delta f[\bar{t}]$; using this denotation we shall express the difference operation by the formula

$$\Delta f[\bar{t}] = f[\bar{t} + 1] - f[\bar{t}]. \tag{4}$$

The result of the above operation — that is the function $\Delta f[\bar{t}]$ is called the "difference of the first order" of the function $f[\bar{t}]$.

The difference operation may be performed several times — for example two or three times. The result of the k-fold difference operation is denoted by the symbol $\Delta^k f[\bar{t}]$, and is called the „difference of the k-th order" of the function $f[\bar{t}]$. This operation is determined by the recurrent formula

$$\Delta^k f[\bar{t}] = \Delta^{k-1} f[\bar{t}+1] - \Delta^{k-1} f[\bar{t}]. \tag{5}$$

Thus, for instance, the difference of the second order of the function $f[\bar{t}]$ is the following step function $\Delta^2 f[t]$:

$$\Delta^2 f[\bar{t}] = \Delta f[\bar{t}+1] - \Delta f[\bar{t}] = f[\bar{t}+2] - 2f[\bar{t}+1] + f[\bar{t}] \tag{6}$$

We may easily observe that in accordance with Definition (5), the difference $\Delta^k f[\bar{t}]$ can always be represented in the form of a linear combination of the operations $T^k f[\bar{t}]$. Namely, by means of induction we can deduce the followng formula

$$\Delta^k f[\bar{t}] = \sum_{\nu=0}^{k} (-1)^\nu \frac{k!}{\nu!\,(k-\nu)!} T^{k-\nu} f[\bar{t}], \tag{7}$$

where, in view of Formula (3)

$$T^{k-\nu} f[\bar{t}] = f[\bar{t}+k-\nu].$$

Accordingly the difference does not introduce any new analytical consequences as regards the displacement operation already presented, since finally it can always be reduced to the displacement operation.

The operation of *summing* sets into correspondence the step function $f[\bar{t}]$ with the function $\sum_{m=0}^{[\bar{t}]} f(m)$, where the symbol $[\bar{t}]$ denotes the function "entire $\bar{t}$". If, for instance, the summing operation is denoted by the symbol $\sigma f[\bar{t}]$, then we obtain the formula

$$\sigma f[\bar{t}] = \sum_{m=0}^{[\bar{t}]} f(m). \tag{8}$$

Summing in the domain of step function plays a role similar to that of integration in the analysis of continuous functions, and the difference operation — the role similar to differentiation.

Note that the operations of displacement, difference and summing described above are additive and homogeneous — that is, they satisfy the following relations

$$\Phi\{f_1[t]+f_2[t]\}=\Phi f_1[t]+\Phi f_2[t],$$
$$\Phi\{Cf[t]\}=C\Phi f[t], \tag{9}$$

where Φ is the symbol of operation, and C is an arbitrary constant.

The relation between the step function $f[\bar{t}]$ and its differences $\Delta f[\bar{t}]$, $\Delta^2 f[\bar{t}], \ldots \Delta^k f[\bar{t}]$ is called the difference equation. Of particular importance to applications is the linear difference equation with constant coefficients. This equation has the following form

$$a_0 \Delta^k f[\bar{t}] + a_1 \Delta^{k-1} f[\bar{t}] + \ldots + a_k f[\bar{t}] = \varphi[\bar{t}], \tag{10a}$$

where $a_0, a_1, \ldots a_k$ are complex numbers and $\varphi[\bar{t}]$ is a step function given in advance. Using the relations (7) we can write every difference equation also in the form presented below, which is convenient in applications

$$b_0 f[\bar{t}+k] + b_1 f[\bar{t}+k-1] + \ldots + b_k f[\bar{t}] = \varphi[\bar{t}] \tag{10b}$$

The classical method of solving difference equations is very close to the classical method of solving differential equations. To get acquainted with these methods, the reader is referred to the literature — for example, to the papers [1] and [2].

The terminology concerning difference equations is also very close to that used in the problems of differential equations. Thus, for instance, the Eq. (10b) in which $\varphi[\bar{t}] \neq 0$, $b_0 \neq 0$ and $b_k \neq 0$ is called a „nonhomogeneous equation of the k-th order". In the case in which $\varphi[\bar{t}] = 0$, this equation will be called a homogeneous equation.

It is worth noting that the order of a difference equation is not always equal to the order of the largest difference $\Delta^k f[\bar{t}]$ if the equation is written in the form of (10a), whereas the form of (10b) does not entail this inconvenience.

The problems of operations performed on step functions, and in particular the problem of difference equations, are dealt with in a special branch of mathematics which is called the "difference calculus".

Below are given simple examples illustrating the manner of determining differences and sums of step functions.

Example 1. Let us calculate the differences of the step function

$$f(n) = n$$

By virtue of Formula (4), the difference of the first order of the function $f(n) = n$ is equal to unity, since we have

$$\Delta f(n) = n + 1 - n = 1$$

The differences of higher orders are equal to zero

$$\Delta^2 f(n) = \Delta^3 f(n) = \ldots = 0$$

Example 2. Let us calculate the difference of the step function

$$f(n) = e^n$$

For the difference of the first order we obtain the formula

$$\Delta f(n) = e^{n+1} - e^n = e^n (e - 1)$$

and for the differences of higher orders

$$\Delta^2 f(n) = \Delta f(n+1) - \Delta f(n) = (e-1)^2 e^n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta^k f(n) = (e-1)^k e^n.$$

Example 3. Let us calculate the sum of the step function

$$f(n) = a^n.$$

We shall obtain

$$\sigma f(n) = \sum_{m=0}^{n} a^m = \frac{a^{n-1} - 1}{a - 1}$$

3. LAPLACE TRANSFORMATION OF STEP FUNCTIONS

As we know, the Laplace transformation is called the following equation

$$F(s) = \int_0^\infty f(t) e^{-st} \, dt \tag{11}$$

which sets into correspondence the function $f(t)$ of the real variable t with the complex function $F(s)$ of the complex variable s. We assume that the function $f(t)$ ensures an absolute convergence of the improper integral (11) in the semispace $\mathcal{R}_e\, s > c$, where c is a number chosen for $f(t)$. We then assume the existence of the limit

$$\lim_{T\to\infty} \int_0^T |f(t)| e^{-\sigma t} \, dt \tag{12}$$

where $\sigma > c$.

The Laplace transformation is abbreviated by the symbol

$$F(s) = \mathcal{L}\{f(t)\} \tag{13}$$

Let us assume that we have a given step function $f[\bar{t}]$ determined for $t \geqslant 0$, and calculate its Laplace transformation. According to (11) we have

$$F(s) - \int_0^\infty f[\bar{t}]\, e^{-s\bar{t}}\, d\bar{t} = \sum_{n=0}^{\infty} \int_n^{n+1} f[\bar{t}]\, e^{-s\bar{t}}\, d\bar{t} = \sum_{n=0}^{\infty} f(n) \int_n^{n+1} e^{-s\bar{t}}\, d\bar{t} =$$

$$= \sum_{n=0}^{\infty} f(n)\, e^{-sn} \frac{1}{s} (1 - e^{s})$$

and after introducing the denotation

$$\eta = \frac{1}{s} (1 - e^{-s}) \tag{14}$$

$$F(s) = \mathcal{L}\{f[\bar{t}]\} = \eta \sum_{n=0}^{\infty} f(n)\, e^{-sn} \tag{15}$$

The series (15) for $\mathcal{R}_e\, s > c$ is evidently absolutely convergent for every function $f[\bar{t}]$ satisfying the condition (12). The number c will be called the „abscissa of convergence" of the series (15).

It may be noted that the quotient $\frac{\mathcal{L}\{f[\bar{t}]\}}{\eta}$ is dependent on the complex variable s only through the exponential functions e^{-sn}, since we have

$$\frac{F(s)}{\eta} = \frac{\mathcal{L}\{f(\bar{t})\}}{\eta} = \sum_{n=0}^{\infty} f(n)\, e^{-sn} \tag{16}$$

Thus if we introduce a new variable z

$$z = e^{s} \tag{17}$$

and moreover, if we denote

$$F^*(z) = \frac{\mathcal{L}\{f[\bar{t}]\}}{\eta} \tag{18}$$

then the function $F^*(z)$ will be the sum of the following series

$$F^*(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} \tag{19}$$

It is worth while to add that Formula (19) determining the correspondence of the step function $f(n)$ with the function $F^*(z)$ may formally be considered as a new functional transformation. Formula (16) may also be taken as the definition of a certain transformation determined on the set of step functions. Y. Z. Tsypkin calls this transformation a "discrete Laplace transformation" and denotes it by the symbol $D\{f(n)\}$ [1].

Using Formula (16), we can determine the Laplace transformation of step functions without the necessity of calculating the integral (11).

Below are presented simple examples of calculating the functions $F^*(z)$ and $F(s)$ for certain step functions.

Example 1. Let $f[t] = 1(t)$ be a given function; by virtue of (19), we have for $|z| > 1$:

$$F^*(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

and, in view of (16) and (17)

$$F(s) = \mathcal{L}\{1\} = \frac{e^s}{e^s - 1}\eta; \qquad \mathcal{R}e\, s > 0$$

Example 2. Let $f[\bar{t}] = e^{\alpha n}$ be a given step function. We have

$$F^*(z) = \sum_{n=0}^{\infty} e^{\alpha n} z^{-n} = \frac{z}{z - e^{\alpha}}; \qquad |z| > |e^{\alpha}|$$

Hence

$$F(s) = \mathcal{L}\{e^{\alpha n}\} = \frac{e^s}{e^s - e^{\alpha}}\eta; \qquad \mathcal{R}e\, s > \alpha.$$

4. CERTAIN PROPERTIES OF LAPLACE TRANSFORMATION

We shall now provide proofs for more important properties of the Laplace transformation concerning step functions. The knowledge of these properties will be necessary in using the method of the Laplace transformation in the problems of the theory of sampled-data systems.

Property 1. If $f[\bar{t}]$ is a transformable step function, then

$$\mathcal{L}\{f[\bar{t} + k]\} = e^{sk}\mathcal{L}\{f[\bar{t}]\} - \eta e^{sk} \sum_{n=0}^{k-1} f(n)\, e^{-sn} \tag{20}$$

where k is a natural number and $\eta = \frac{1}{s}(1 - e^{-s})$.

Proof. According to (16), we have

$$\frac{\mathcal{L}\{f[\bar{t} + k]\}}{\eta} = \sum_{n=0}^{\infty} f(n + k)\, e^{-sn}$$

If, then, we introduce a new variable of summing $n_1 = n + k$, we shall obtain

$$\frac{\mathcal{L}\{f[\bar{t}+k]\}}{\eta} = \sum_{n_1=k}^{\infty} f(n_1)\, e^{-s(n_1-k)} = e^{sk} \sum_{n_1=0}^{\infty} f(n_1)\, e^{-sn_1} - e^{sk} \sum_{n_1=0}^{k-1} f(n_1)\, e^{-sn_1}.$$

Since the result of summing does not depend on the manner of denoting the variable, then

$$\mathcal{L}\{f[\bar{t}+k]\} = e^{sk}\, \mathcal{L}\{f[\bar{t}]\} - \eta\, e^{sk} \sum_{n=0}^{k-1} f(n)\, e^{-sn}$$

By denoting $e^s = z$, we may write Formula (20) in the form

$$\mathcal{L}\{f[\bar{t}+k]\} = z^k\, \mathcal{L}\{f[\bar{t}]\} - \eta z^k \sum_{n=0}^{k-1} f(n)\, z^{-n} \qquad (20a)$$

In a particular case in which

$$f(0) = f(1) = \ldots = f(k-1) = 0$$

which means that the step function $f[\bar{t}]$ is identically equal to zero in the interval $O \leqslant t < k$, we shall obtain

$$\mathcal{L}\{f[\bar{t}+k]\} = z^k\, \mathcal{L}\{f[\bar{t}]\} \qquad (21)$$

Property 2. If $f[\bar{t}]$ is a trasformable step function, then

$$\mathcal{L}\{\Delta^k f[\bar{t}]\} = (z-1)^k\, \mathcal{L}\{f[\bar{t}]\} - \eta z \sum_{n=0}^{k-1} \Delta^n f[0]\,(z-1)^{k-1-n} \qquad (22)$$

where k is a natural number, $z = e^s$, and $\eta = \frac{1}{s}(1 - e^{-s})$.

Proof. Let us first assume that $k = 1$. Then, in accordance with the definition of the difference $\Delta f[\bar{t}]$ we have

$$\mathcal{L}\{\Delta f[\bar{t}]\} = \mathcal{L}\{f[\bar{t}+1]\} - \mathcal{L}\{f[\bar{t}]\}$$

and on account of Formula (20a)

$$\mathcal{L}\{\Delta f[\bar{t}]\} = z\mathcal{L}\{f[\bar{t}]\} - \eta z f[0] - \mathcal{L}\{f[\bar{t}]\} = (z-1)\,\mathcal{L}\{f[\bar{t}]\} - \eta z f[0] \quad (22a)$$

Assuming in turn that $k = 2$ and taking into consideration that

$$\Delta^2 f[\bar{t}] = \Delta f[\bar{t}+1] - \Delta f[\bar{t}]$$

after performing simple calculations we shall obtain

$$\mathcal{L}\{\Delta^2 f[\bar{t}]\} = (z-1)^2\, \mathcal{L}\{f[\bar{t}]\} - \eta z (z-1) f[0] - \eta z \Delta f[0] \qquad (22b)$$

We have thus proved that Formula (22) holds for the case in which $k = 1$ and $k = 2$. The correctness of Formula (22) may then be proved by induction.

Property 3. If $f_1[t]$ and $f_2[t]$ are two transformable step functions, then

$$\frac{\mathcal{L}\{f_1[\bar{t}]\}\,\mathcal{L}\{f_2[\bar{t}]\}}{\eta} = \mathcal{L}\left\{\sum_{m=0}^{\infty} f_1[m]\, f_1[\bar{t}-m]\right\} \tag{23}$$

where $\eta = \dfrac{1}{s}(1-e^{-s})$.

Proof. By virtue of Formula (16) we have

$$\frac{\mathcal{L}\{f_1[\bar{t}]\}\,\mathcal{L}\{f_2[\bar{t}]\}}{\eta} = \eta \sum_{n=0}^{\infty} f_1(n)\, z^{-n} \sum_{n=0}^{\infty} f_2(n)\, z^{-n}$$

where $z = e^s$.

Thus, if we calculate the product of the series according to Cauchy's formula

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m\, b_{n-m}$$

we shall obtain

$$\eta \sum_{n=0}^{\infty} f_1(n)\, z^{-n} \sum_{n=0}^{\infty} f_2(n)\, z^{-n} = \eta \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{n} f_1(m)\, f_2(n-m) =$$

$$= \eta \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{\infty} f_1(m)\, f_2[\bar{t}-m]$$

and on account of Formula (16)

$$\frac{\mathcal{L}\{f_1[\bar{t}]\}\,\mathcal{L}\{f_2[\bar{t}]\}}{\eta} = \mathcal{L}\left\{\sum_{m=0}^{\infty} f_1(m)\, f_2[\bar{t}-m]\right\}$$

Property 4. If $f[\bar{t}]$ is a transformable step function, and $F(s) = \mathcal{L}\{f[\bar{t}]\}$ then

$$\mathcal{L}\{n^k f[\bar{t}]\} = (-1)^k\, \eta \frac{d^k}{ds^k}\left[\frac{F(s)}{\eta}\right] \tag{24}$$

where $\eta = \dfrac{1}{s}(1-e^{-s})$ and k is a natural number.

Proof. In fact, since

$$\frac{F(s)}{\eta} = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$$

then

$$\frac{d^k}{ds^k}\left[\frac{F(s)}{\eta}\right] = \sum_{n=0}^{\infty}(-n)^k f(n) e^{-sn} = (-1)^k \frac{1}{\eta} \mathcal{L}\{n^k f[\bar{t}]\}$$

and hence follows immediately the relation (24). Expressing the relation (24) by means of the function $F^*(e^s)$ determined by the dependence (19), we shall obtain a simpler formula, namely

$$\frac{\mathcal{L}\{n^k f[\bar{t}]\}}{\eta} = (-1)^k \frac{d^k}{ds^k} F^*(e^s) \tag{24a}$$

Property 5. If $f[\bar{t}]$ is a transformable step function and $F(s) = \mathcal{L}\{f[\bar{t}]$, then

$$\lim_{s\to 0} \frac{F(s)}{\eta} = \sum_{n=0}^{\infty} f(n) \tag{25}$$

where $\eta = \frac{1}{s}(1 - e^{-s})$.

Proof. In fact, since

$$\frac{Fs}{\eta} = \sum_{n=0}^{\infty} f(n) e^{-sn}$$

then

$$\lim_{s\to 0} \sum_{n=0}^{\infty} f(n) e^{-sn} = \sum_{n=0}^{\infty} f(n).$$

The relation (25) may also be expressed by means of the function $F^*(e^s)$ determined by Formula (19). We shall then obtain

$$\lim_{s\to 0} F^*(e^s) = \sum_{n=0}^{\infty} f(n)$$

Note that the series standing on the right-hand side of Equations (25) and (25a) determines the field comprised between the plot of the step function and the abscissa axis. This series is evidently convergent only in the case in which this field is limited; this takes place when the abscissa of convergence c in the series (16) is a negative number.

5. RELATIONS BETWEEN THE IMAGE OF STEP FUNCTIONS AND THE FOURIER SERIES

Let us assume that the abscissa of convergence c of the step function $f[\bar{t}]$ is a negative number. This means that the function $f[\bar{t}]$ asymptotlically tends to zero with increase in the variable $\bar{t}$ — that is, it satisfies the condition

$$\lim_{\bar{t}\to\infty} f[\bar{t}] = 0$$

In this case, in Formula (19) determining the function

$$F^*(e^s) = \frac{F(s)}{\eta} = \frac{\mathfrak{L}\{f[\bar{t}]\}}{\eta}$$

we may assume that $s = j\omega$. The function $F^*(e^{j\omega})$ will then be the sum of the following series

$$F^*(e^{j\omega}) = \sum_{n=0}^{\infty} f(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} f(n) \cos \omega n - j \sum_{n=0}^{\infty} f(n) \sin \omega n. \qquad (26)$$

It is worth noting that the right-hand side of Formula (26) is simply a Fourier series with the real coefficients $f(n)$. Hence follow the interesting conclusions stated below.

First of all, it follows from Formula (26) that the function $F^*(e^{j\varepsilon})$ is a periodic function of the variable ω. Moreover, we immediately state that the values $f(n)$ of the step function, as coefficients of the Fourier series, are related with the function $F^*(e^{j\omega})$ by means of the formula

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F^*(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad (27)$$

The expression $F^*(e^{j\omega})$, as a complex function, can of course be written in the form of the real and the imaginary component. If, then, we denote the real component of the function $F^*(e^{j\omega})$ by $A^*(\omega)$, and the imaginary component by $jB^*(\omega)$, then

$$F^*(e^{j\omega}) = A^*(\omega) + jB^*(\omega)$$

and by virtue of (26)

$$A^*(\omega) = \sum_{n-0}^{\infty} f(n) \cos \omega n,$$
$$B^*(\omega) = - \sum_{n=0}^{\infty} f(n) \sin \omega n. \qquad (28)$$

Let us now consider Formula (27). Substituting in this formula $e^{j\omega n} = \cos \omega n + j \sin \omega n$ and $F^*(e^{j\omega}) = A^*(\omega) + jB^*(\omega)$ we shall obtain

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [A^*(\omega) \cos \omega n - B^*(\omega) \sin \omega n]\, d\omega +$$

$$+ j\frac{1}{2\pi}\int_{-\pi}^{\pi}[A^*(\omega)\sin\omega n + B^*(\omega)\cos\omega n]\,d\omega .$$

Since A^*, as a function $\cos\omega$, is an even function, and B^*, as a function ω, is an odd function, then the integrand in the first integral is an even function of ω, while in the second integral — an odd function of ω. The value of the second integral is then equal to zero, and hence

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[A^*(\omega\cos\omega n - B^*(\omega)\sin\omega n]\,d\omega =$$

$$= \frac{1}{\pi}\int_{0}^{\pi}[A^*(\omega)\cos\omega n - B^*(\omega)\sin\omega n]\,d\omega . \tag{29}$$

If $f(n) = 0$ for $n < 0$, Formula (29) is further simplified, and we obtain

$$\left.\begin{aligned} f(n) &= \frac{2}{\pi}\int_{0}^{\pi} A^*(\omega)\cos\omega n d\omega , \\ f(n) &= -\frac{2}{\pi}\int_{0}^{\pi} B^*(\omega)\sin\omega n d\omega . \end{aligned}\right\} \tag{30}$$

In fact, substituting $-n$ for n in Formula (29), we shall obtain

$$\frac{1}{\pi}\int_{0}^{\pi}[A^*(\omega)\cos\omega n + B^*(\omega)\sin\omega n]\,d\omega = 0, \tag{31}$$

and hence, after taking into consideration (31) in (29), we arrive immediately at Formula (30).

PART II

OUTLINE OF THE THEORY OF SAMPLED-DATA SYSTEMS

1. NOTION OF A SAMPLED-DATA SYSTEM

The sampled-data system is a system consisting of two elements connected in series namely, of what is called a sampler, and a linear dynamical system (Fig. 2).

The task of the sampler is to transform the input signal $x_1(t)$ usually representing a continuous function into the signal $x_1(n)$ in the form of rectangular pulses. We assume that the pulses generated by the sampler at the moment of their appearance are proportional to the function of the input signal $x_1(t)$. In other words, we assume that the heights of the pulses $x_1(n)$ are modulated by the function $x_1(t)$. Moreover, we assume that the period of the occurrence of pulses and the width of pulses are constant quantities which do not change in time (Fig. 3). The sampler performing the transformation of the signal will then be characterized by three independent parameters namely, the amplification k, the period of generated pulses T, and the width of pulses γT; $\gamma \leqslant 1$. These three parameters fully characterize the proporties of a sampler.

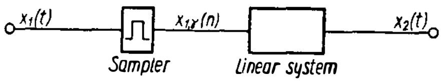


Fig. 2. Sampled-data system

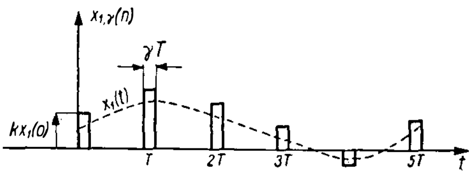


Fig. 3. Pulse signal generated by a sampler

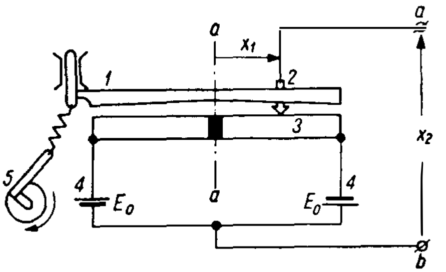


Fig. 4. Operation principle of a mechanical sampler of the first type. (1) bar, (2) indicating needle of the galvanometer, (3) winding of the resistor, (4) batteries of voltage E, (5) driving mechanism

The construction of samplers may vary considerably as to kind. The sampler may be, for instance, a key, an electronic relay, etc. In control systems, an electromechanical sampler is usually applied, constructed as shown at Fig. 4. The action of such a sampler is as follows. The bar *1* is set into a periodic vibrating motion (up and down) by means of a special mechanism *5*: owing to this motion, in presses, at certain determined intervals of time, the indicating needle *2*, of the galvanometer to the winding of the resistor *3*. Two batteries of voltage E are applied to the terminals of this resistor. The input signal $x_1(t)$ of the sampler is the deflection of the indicating needle *2* from the central point of the resistor *3*. The output signal is the voltage $x_1(n)$ measured at the terminals, a and b. If the resistor is wound uniformly, the heights of the signal pulses, $x_1(n)$ are proportional to the deflection of the indicating needle *2*, from the central point of the resistor, and thus the heights of pulses are proportional to the signal $x_1(t)$.

A sampler so constructed is evidently simultaneously an amplifier. Since the energy of the output signal $x_2(n)$ may considerably exceed the energy of the input signal $x_1(n)$.

Every dynamical system under the influence of a signal in the form of a sequence of pulses, even though it possesses no sampler, may of course also be reckoned in the category of sampled-data systems and considered in terms of the theory presented in this paper, for it is always possible to treat a signal in the form of pulses as an output signal of a certain fictitious sampler. The input signal of such a sampler may have an arbitrary shape as long as at moments of the occurrence of pulses the signal is proportional to the height of pulses.

It is worth noting that sampled-data systems have various applications beyond control engineering. They are encountered, for instance, in radio engineering, radiolocation and television.

2. EQUATIONS OF A SAMPLED-DATA SYSTEM

The linear element of a sampled-data system may be any linear dynamical system. In a particular case, the linear element may be an electrical four-pole, a mechanical system, an electromechanical system, etc. In the present discussion, we shall not be concerned with the essential properties and the structure of the linear element, but shall investigate its properties in terms of well-known general laws governing linear dynamical systems. Moreover, we shall confine ourselves to investigating systems whose dynamical properties can be described by means of ordinary differential equations — that is, to systems constructed of lumped elements. We knew from the theory of dynamical systems, that the transmission properties — that is, the "capabilities of transmitting signals" of a dynamical system — can uniquely be determined from the knowledge of a certain rational function

$$K(s) = \frac{P(s)}{Q(s)} \tag{32}$$

which is the ratio of the Laplace transformations of the output signal to the input signal. This function is called the transfer function of the system.

The transfer function $K(s)$ of a system can be determined in a simple manner by well-known methods; for example, it may be calculated from the differential equations of the system by means of the substitution of the complex variable s^k for the symbol of differentiation $\frac{d^k}{dt^k}$. It can be proved that, for every real dynamical system, the degree of the polynomial $P(s)$ cannot exceed the degree of the polynomial $Q(s)$ of the transfer function $K(s)$.

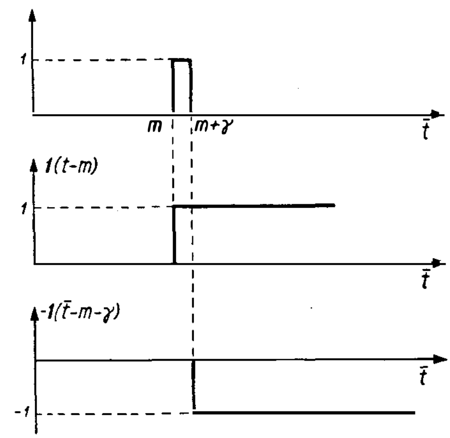


Fig. 5. Representation of a rectangular pulse by means of the difference of two displaced unit-step functions

The transmission properties of a dynamical system can also be determined by means of the so-called unit-step response of the system that is, by means of the function of the output sygnal actuated by the input excitation in the form of a unit-step function. The unit-step response is usually denoted by the symbol $h(t)$. It is related to the transfer function $K(s)$ of the system in the following formula

$$h(t) = \mathcal{L}^{-1}\left\{\frac{K(s)}{s}\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{K(s)}{s} e^{st}\, ds \tag{33}$$

If we know the function $h(t)$, and thus also the function $h(\bar{t}) = h\left(\frac{t}{T}\right)$, then we can easily determine the response of the system to the excitation in the form of a rectangular pulse. Namely, toking into consideration that a rectangular pulse with a unit height and with duration from $\bar{t}_1 = m$ to $\bar{t}_2 = m + \gamma$, can be considered as the difference of two displaced unit-step functions (Fig. 5), we shall easily find that the responses of the system to such a pulse will be equal to

$$\begin{aligned} &h(\bar{t} - m), \text{ for } m \leqslant \bar{t} < m + \gamma; \\ &h_\gamma(\bar{t} - m) = h(\bar{t} - m) - h(\bar{t} - m - \gamma), \text{ for } \bar{t} \geqslant m + \gamma \end{aligned} \tag{34}$$

The first expression determines the response of the system to a rectangular pulse during the duration of the pulse, and the second expression — beyond the duration of the pulse.

Let us now assume that on the system there acts a signal in the form of a sequence of rectangular pulses with identical widths and with heights changing according to the law $kx_1(m)$ (Fig. 6). The output signal

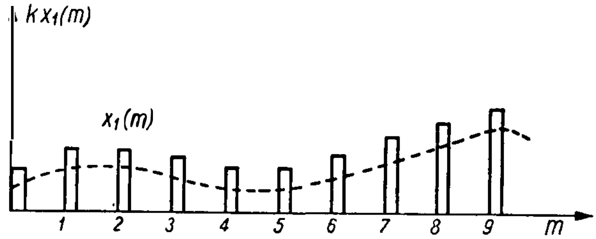


Fig. 6. Waveform at the input of a linear element of a sampled-data system

of the system will then be the sum of all the elementary responses of the system to each pulse having the height $kx_1(m)$ and the width γ. Accordingly, considering the time interval $n \leqslant \bar{t} < n + 1$, we shall obtain

$$\left.\begin{array}{l} x_2(\bar{t}) = k \sum\limits_{m=0}^{n-1} x_1(m) h_\gamma(\bar{t} - m) + kx_1(n) h(\bar{t} - n) \\ \text{for } n \leqslant \bar{t} < n + \gamma \\ \quad \text{and} \\ x_2(\bar{t}) = k \sum\limits_{m=0}^{n} x_1(m) h_\gamma(\bar{t} - m) \\ \text{for } n + \gamma \leqslant \bar{t} < n + 1. \end{array}\right\} \quad (35)$$

The function of the output signal $x_2(\bar{t})$ is then determined by two independent analytical formulae, the first of which (35) defines the signal $x_2(\bar{t})$ in the time intervals corresponding to the duration periods of the exciting pulses.

It should be emphasized that the existence of two independent formulae determining the output signal $x_2(\bar{t})$ is a feature characteristic of every sampled-data system. This fact entails — as we shall see later — the necessity of introducing also two independent frequency charakteristics of a sampled-data system.

Formulae (35) can also be written in a somewhat different form. Namely, introducing the denotation

$$\bar{t} = \frac{t}{T} = n + \frac{\Delta t}{T} = n + \varepsilon$$

we shall obtain

$$\left.\begin{aligned} & x_2(n+\varepsilon) = k\sum_{m=0}^{n-1} x_1(m)\, h_\gamma(n+\varepsilon-m) + kx_1(n)\, h(\varepsilon) \\ & \text{for } n \leqslant n+\varepsilon < n+\gamma \\ & \qquad \text{and} \\ & x_2(n+s) = k\sum_{m=0}^{n} x_1(m)\, h_\gamma(n+\varepsilon-m) \\ & \text{for } n+\gamma \leqslant n+\varepsilon < n+1 . \end{aligned}\right\} \quad (36)$$

The expressions (36) thus obtained determine — for a given value of ε — the relations between the values $x_2(\bar{t})$ at moments when they are equi-distant from one another: $\bar{t} = \varepsilon$, $\bar{t} = 1 + \varepsilon$, $\bar{t} = 2 + \varepsilon, \ldots$ The values of the function $x_2(\bar{t})$ corresponding to the given walue of ε can, then, be replaced by the step function $x_2(n, \varepsilon)$ being depedent on the parameter ε. Taking into consideration that $h_\gamma(\bar{t}) = 0$ for $\bar{t} < 0$ — that is, for $n > m$ — we may extend the limits of summing from n to ∞.

Accordingly, Formuale (36) can be written as

$$\left.\begin{aligned} x_2(n,\varepsilon) &= k\sum_{m=0}^{\infty}{}' x_1(m)\, h_\gamma(n-m,\varepsilon) \qquad 0 \leqslant \varepsilon < \gamma , \\ x_2(n,\varepsilon) &= k\sum_{m=0}^{\infty} x_1(m)\, h_\gamma(n-m,\varepsilon) \qquad \gamma \leqslant \varepsilon < 1 , \end{aligned}\right\} \quad (37)$$

where the comma by the sign of the sum in the first of the formulae (37) indicates that for $n = m$ the expression $h_\gamma(n - m, \varepsilon) = h_\gamma(O, \varepsilon)$ should be replaced by $h(O, \varepsilon)$.

3. PULSE TRANSFER FUNCTIONS

Transmission properties of a linear system can always be determined by means of the following function $K(s)$

$$K(s) = \frac{\mathcal{L}\{x_2(t)\}}{\mathcal{L}\{x_1(t)\}} \quad (38)$$

where x_2 and x_1 are the input and output signals of the linear system, respectively, and L denotes the integral Laplace transformation. The question now arises as to whether it would be possible to determine, similarly by means of a certain function, the capability for transmitting signals by a sampled-data linear system. The answer is positive.

In order to explain this problem, it should first be noted that any continuous function of a real variable t can uniquely be determined by means of a set of step functions dependent on a real parameter. Evidently, if we take into consideration, for example, the function $f(\bar{t})$, where $\bar{t} = \frac{t}{T}$ is a real variable, then the set of step functions $f[\bar{t}, \varepsilon]$ dependent on the parameter $\in \varepsilon$ [0,1] and formed from the function $f(\bar{t})$, uniquely determines the function $f(\bar{t})$. This may be expressed as

$$f(\bar{t}) \sim f[\bar{t}, \varepsilon] \tag{39}$$

Using the representation of the continuous function by means of a set of step functions, we may define the pulse-transfer function as follows

$$K^*(e^s, \varepsilon) = \frac{\mathcal{L}\{x_2[\bar{t}, \varepsilon]\}}{\mathcal{L}\{x_1[\bar{t}, \varepsilon]\}}; \qquad 0 \leqslant \varepsilon < 1 \tag{40}$$

where $x_2[\bar{t}, \varepsilon]$ is a step function dependent on the real parameter ε which determines the output signal $x_2(\bar{t})$ of the sampled-data system; $x_1[\bar{t}]$ is a step function formed from the function $x_1(t)$ of the input signal (Fig. 5).

The function $K^*(e^s, \varepsilon)$ has a simple physical interpretation. It can be shown that $K^*(e^s, \varepsilon)$ is the Laplace transformation of the function k $h_\gamma[\bar{t}, \varepsilon]$ divided by the transformation unit $\eta = \frac{1}{s}(1 - e^{-s})$, where $h_\gamma[\bar{t}, \varepsilon]$ is the time response of the linear system to the exciting rectangular pulse width γT and height equal to unity, and k is the amplification of the sampler. Thus we obtain

$$K^*(e^s, \varepsilon) = \frac{\mathcal{L}\{kh_\gamma[\bar{t}, \varepsilon]\}}{\eta} \tag{41}$$

In fact, according to (40) we may write

$$\frac{1}{\eta}\mathcal{L}\{kh_\gamma[\bar{t}, \varepsilon]\}\,\mathcal{L}\{x_1[\bar{t}]\} = \mathcal{L}\{x_2[\bar{t}, \varepsilon]\}$$

Hence, as the result of the step function convolution, we obtain

$$\mathcal{L}\left\{\sum_{m=0}^{\infty} x_1[m]\,kh_\gamma[\bar{t} - m, \varepsilon]\right\} = \mathcal{L}\{x_2[\bar{t}, \varepsilon]\}$$

and

$$x_2[\bar{t}, \varepsilon] = \sum_{m=0}^{\infty} x_1[m]\,kh_\gamma[\bar{t} - m, \varepsilon] \tag{42}$$

or

$$x_2(\bar{t}) = k \sum_{m=0}^{\infty} x_1(m)\, h_\gamma(\bar{t} - m)$$

However, the relation (42) determines the output signal of the system in the case in which $h(t)$ is the response to the rectangular pulse with height 1 and width T (by virtue of Formulae 35) (Fig. 6). Accordingly, the expression $h(t)$, is in fact the time response of the system to excitation in the form of the rectangular pulse determined above.

Let us take into consideration a formula determining the Laplace transformation of the step function

$$\mathcal{L}\{f(t)\} = \eta \sum_{n=0}^{\infty} f(n)\, e^{-ns} \tag{43}$$

where

$$\eta = \frac{1}{s}(1 - e^{-s})$$

In terms of the above formula, the relation (41) may be written as an infinite sum of exponential functions. Let $K_{\text{I}}^{*}(e^s, \varepsilon)$ denote the function $K^*(e^s, \varepsilon)$ in the interval $\sigma \leqslant \varepsilon < \gamma$ and $K_{\text{II}}^{*}(e^s, \varepsilon)$ denote this function in the interval $\gamma \leqslant \varepsilon < 1$. Thus we obtain

$$\begin{aligned} K_{\text{I}}^{*}(e^s, \varepsilon) &= kh[0, \varepsilon] + k \sum_{n=1}^{\infty} k_\gamma[n, \varepsilon]\, e^{-ns}; \qquad 0 \leqslant \varepsilon < \gamma \\ K_{\text{II}}^{*}(e^s, \varepsilon) &= k \sum_{n=0}^{\infty} h_\gamma[n, \varepsilon]\, e^{-ns}; \qquad \gamma \leqslant \varepsilon < 1 \end{aligned} \tag{44}$$

Formulae (44), which determine the function K_{I}^{*} and K_{II}^{*} differ from each other by the component $h[0, \varepsilon]$, which is the time response to the excitation of the unit step function. This is so, because during the first moments — that is, in the interval $0 \leqslant \varepsilon < \gamma$, in which the first rectangular pulse is not yet finished — the system reacts in such a manner as it would react if the unit step function were an excitation, not the pulse.

Formulae (44) may be used for practical calculations of the pulse-transfer functions K_{I}^{*} and K_{II}^{*}. The function $h_\gamma[n, \varepsilon] = h(\bar{t}) = h\left(\frac{t}{T}\right)$, which occurs in the formulae (44), can be evaluated from the transfer function of the system

$$K(s) = \frac{\mathcal{L}\{x_2(t)\}}{\mathcal{L}\{x_1(t)\}} = \frac{P(s)}{Q(s)}$$

When the function $h_\gamma[n, \varepsilon]$ is found, we may calculate K_{I}^{*} and K_{II}^{*} in a simple manner by summing Formulae (44).

The method of calculating the pulse-transfer functions K_{I}^* and K_{II}^* for the case of systems with lumped elements, reduces to finding the time response — for example, in terms of the formula

$$h(t) = \frac{P(o)}{Q(o)} + \sum_{\nu=1}^{l} \frac{P(s_\nu)}{Q(s_\nu)} e^{s_\nu t} \tag{45}$$

and to determining the function $h\left(\frac{t}{T}\right) = h(n + \varepsilon)$ and $h_\gamma(\mathrm{n}, \varepsilon) = h(n + \varepsilon) - h(n + \varepsilon) - \gamma$, and finally to calculating the sums of the series (44).

Similarly, for any realizable transfer function $K(s)$, we can calculate the corresponding pulse-transfer function K_{I}^* and K_{II}^*. The results of calculations for these functions are appended in Table II in the Annex. In this Table, the function K_ε^* is also given, corresponding to the case of very narrow pulses — that is, if $\gamma \ll 1$.

Further, we shall prove that the function K^* is also a pulse-transfer function of the system with a modulation of the width of pulses — that is, for sampled-data systems of a different type. However, the amplification coefficient of the sampler should in this case be substituted for the width coefficient $\varkappa$ of pulses.

This fact is of great practical significance, since it makes it possible to apply the theory presented in this paper to sampled-data control systems with a variable width of pulses.

Of course it should be possible to confine the pulse-transfer function to the case of very narrow pulses $\gamma \ll 1$, since we can obtain every pulse which has a different form from the Dirac $\delta(t)$ pulse, by acting on the input of the proper forming element by means of the function $\mathbf{1}(t)$. Thus the considerations could be limited merely to a sequence of Dirac pulses, as has been done by several authors. However, the differentiation between the pulse-transfer functions K_{I}^* and K_{II}^* simplifies in many cases the analysis of systems.

Taking into consideration Formula (40), the pulse-transfer function K^* entirely determines the capability of the system of transmitting pulse signals, since — knowing $K^*(e^s, \varepsilon)$ and $X_1[\bar{t}]$ — we can always calculate the waveform of the signal $X_2[\bar{t}, \varepsilon] \sim X_2(\bar{t})$ from the Formula

$$\mathcal{L}\{X_2[t, \varepsilon]\} = K^*(e^s, \varepsilon)\, \mathcal{L}\{X_1[t]\} \tag{46}$$

The Laplace transform pairs of unit-step functions, which are appended in Table I, are helpful in the calculations.

Introducing the notation

$$z = e^s \tag{47}$$

we may write the function $K(e^s, \varepsilon)$ in the form

$$K^*(z, \varepsilon) \tag{48}$$

4. CHARACTERISTICS OF SAMPLED-DATA SYSTEMS

4.1. Time characteristic

The properties of sampled-data systems (similarly to those continuous of dynamical systems) may be expressed by means of time characteristic or frequency characteristic.

The time characteristic of a sampled-data system is determined by the response of the sampled-data system to a unit step excitation. The frequency characteristic of a sampled-data system is determined by the steady-state component of the response of the system to an excitation by means of a sinusoidal function.

The first of the above characteristics is treated as a time function, and the second — as a frequency function.

The time characteristic will be denoted by the symbol $b[n, \varepsilon]$. This characteristic may be determined directly from the equations of a sampled-data system

$$\mathcal{L}\{x_2[n, \varepsilon]\} = K_i^*(z, \varepsilon)\mathcal{L}\{x_1[n]\} \tag{46}$$

with the assumption that

$$\mathcal{L}\{x_1[n]\} = \mathcal{L}\{1[n]\} = \frac{z}{z-1}\eta$$

thus we have

$$\mathcal{L}\{b[n, \varepsilon]\} = K_i^*(z, \varepsilon)\frac{z}{z-1}\eta$$

We shall prove that the time characteristic $b[n, \varepsilon]$ determines uniquely the transmission properties of a sampled-data system — that is, from the knowledge of the characteristic $b[n, \varepsilon]$ is always possible to determine the output signal $x_2[n, \varepsilon]$ in the case of the excitation $x_1[n]$.

Let us assume that the input signal of a sampled-data system is the function $x_1(t)$. Then at the output of the sampler we shall obtain a signal in the form of a sequence of rectangular pulses. The responses of the system to the particular pulses are defined by the following formulae

$$x_1[0][b[n, \varepsilon] - b[n-1, \varepsilon]] = x_1[0]\Delta b[n-1, \varepsilon],$$

$$x_1[1]\Delta b[n-2, \varepsilon],$$

$$x_1[2]\Delta b[n-3, \varepsilon],$$

. . . .

$$x_1[n][b[0, \varepsilon] - 0] = x_1[n]b[0, \varepsilon]$$

Summing up these formulae, we shall obtain the output signal $x_2[n, \varepsilon]$

$$x_2[n, \varepsilon] = x_1[n]\, b[0, \varepsilon] + \sum_{m=1}^{n} x_1[m-1]\, \Delta b[n-m, \varepsilon] \tag{47}$$

Formula (47) is analogous to a formula well-known in operator calculus — namely

$$x_2(t) = x_1(t)\, h(0) + \int_0^t x_1(\tau)\, h'(t-\tau)\, d\tau \tag{48}$$

which determines the output signal $x_2(t)$ of a dynamical system in terms of a given unit-step response $h(t)$.

After simple transformations, Formula (47) may also be written in the equivalent form

$$x_2[n, \varepsilon] = x_1[0]\, b[n, \varepsilon] + \sum_{m=1}^{n} \Delta x_1[m-1]\, b[n-m, \varepsilon] \tag{49}$$

4.2. Frequency characteristic

Let $\bar{\omega}$ be the symbol denoting what is called the "dimensionless angular frequency"

$$\bar{\omega} = \omega T \tag{50}$$

where T is the period of the occurrence of rectangular pulses.

The frequency characteristic of a sampled-data system will be called the following function $M(j\bar{\omega}, \varepsilon)$

$$M^*(j\bar{\omega}, \varepsilon) = \left. \frac{\mathcal{L}\{x_1[\bar{t}, \varepsilon]\}}{\mathcal{L}\{x_1[\bar{t}]\}} \right|_{s=j\bar{\omega}} \tag{51}$$

In view of this definition and Formula (41), we arrive at

$$M^*(j\bar{\omega}, \varepsilon) = K_i^*(e^{j\bar{\omega}}, \varepsilon) \tag{52}$$

The frequency characteristic of a sampled-data system is then obtained from the pulse-transfer function $K_i^*(e^s, \varepsilon)$ of the system by substituting for the complex variable s the variable $j\bar{\omega} = j\omega T$.

We shall now prove that the frequency characteristic $M(j\omega, \varepsilon)$ of a stable sampled-data system is equal to the ratio of the steady-state component $x_{2ust}[n, \varepsilon]$, of the response of the system to a harmonic excitation, to the input signal $x_1[n]$, and thus the following equality is satisfied

$$M^*(j\bar{\omega}, \varepsilon) = \frac{x_{2\,steady}[n, \varepsilon]}{x_1[n]} \tag{53}$$

where

$$x_1[n] = e^{j\bar{\omega} n}$$

Let us next consider the equation

$$\mathcal{L}\{x_2[n,\varepsilon]\} = K_i^*(e^s,\varepsilon)\,\mathcal{L}\{e^{j\bar{\omega}n}\} \tag{54}$$

determining the response of a sampled-data system to a harmonic excitation.

Taking into consideration in Equation (54)

$$\mathcal{L}\{e^{j\omega n}\} = \frac{e^s}{e^s - e^{j\omega}}\,\eta \tag{55}$$

we shall obtain

$$\mathcal{L}\{x_2[n,\varepsilon]\} = K_i^*(e^s,\varepsilon)\,\frac{e^s}{e^s - e^{j\bar{\omega}}}\,\eta = \frac{P_i^*(e^s,\varepsilon)}{Q^*(e^s)}\cdot\frac{e^s}{e^s - e^{j\bar{\omega}}}\,\eta \tag{56}$$

where

$$K_i^*(e^s,\varepsilon) = \frac{P_i^*(e^s,s)}{Q^*(e^s)}$$

and the functions $P_i^*(e^s,\varepsilon)$ and $Q^*(e^s)$ are polynomials of e^s.

Expanding the rational function standing at the right-hand side of Equation (56) into simple fractions, we shall obtain in the case of single poles

$$\mathcal{L}\{x_2[n,\varepsilon]\} = A_0(\varepsilon)\,\frac{e^s}{e^s - e^{j\omega}}\,\eta + \sum_{\nu=1}^{l} A_\nu(\varepsilon)\,\frac{e^s}{e^s - e^{s_\nu}}\,\eta \tag{57}$$

where e^{s_ν} are the zeroes of the polynomial $Q^*(e^s)$, and l is the degree of the polynomial $Q^*(e^s)$.

Taking now into account the following obvious equality

$$\mathcal{L}^{-1}\left\{\frac{e^s}{e^s - e^\alpha}\right\} = e^{\alpha n} \tag{58}$$

we shall arrive at

$$x_2[n,\varepsilon] = A_0(\varepsilon)\,e^{j\omega n} + \sum_{\nu=1}^{l} A_\nu(\varepsilon)\,e^{s_\nu n} \tag{59}$$

If the sampled-data system is a stable system (which is always the case when the linear element of the system is stable), then all the numbers s_ν — as the poles of the transfer $K(s)$ of the linear element — will have negative real parts

$$\mathcal{R}e\, s_\nu < 0 \tag{60}$$

Hence it follows that the term appearing ta the right-hand side of Formula (59) represents a "decaying" function — thas is, a function satisfying the condition

$$\lim_{n\to\infty} \sum_{\nu=1}^{l} A_\nu(\varepsilon)\, e^{s_\nu n} = 0 \tag{61}$$

In the case in which the pulse transfer function

$$K_i^* = (e^s, \varepsilon) = \frac{P_i^*(e^s, \varepsilon)}{Q^*(e^s)}$$

has multiple poles, then the second term of Formula (59) contains factors of the form

$$\frac{n(n-1)\dots(n-\mu+1)}{\mu\,!}\, e^{s_\nu (n-\mu)}$$

which for $\mathcal{R}_e\, s_\nu < 0$ evidently also represent decaying functions. Thus the steady-state component of the output signal $x_2[n, \varepsilon]$ — independently of whether the pulse-transfer function $K_i(e^s, \varepsilon)$ has single or multiple poles — is the function $A_0(\varepsilon)\, e^{j\bar{\omega}n}$

$$x_{2ust}[n, \varepsilon] = A_0(\varepsilon)\, e^{j\bar{\omega}n} \tag{62}$$

$A_0(\varepsilon)$, as the coefficient of expanding a rational function with a single pole, is calculated as follows

$$A_0(\varepsilon) = \lim_{s\to j\omega} \frac{K_i^*(e^s, \varepsilon)}{e^s - e^{j\omega}}\,(e^s - e^{j\bar{\omega}}) = K_i^*(e^{j\omega}, \varepsilon) = M^*(j\bar{\omega}, s) \tag{63}$$

Since the input signal is the waveform represented by the function

$$x_1[n] = e^{j\bar{\omega}n}$$

then from Formulae (62) and (63) follows relation (53)

$$M^*(j\bar{\omega}, \varepsilon) = \frac{x_{2\,steady}[n, \varepsilon]}{x_1[n]}$$

where $x_1[n] = e^{j\bar{\omega}n}$.

Giving the expression (53) a physical interpretation, we may consider that the frequency characteristic $M(j\bar{\omega}, \varepsilon)$ determines the capability of the system for transmitting the "amplitude" and "phase" of a sinusoidal signal. In fact, in the case of applying at the output the excitation

$$x_1[n] = \mathcal{R}_e\, e^{j\omega n}$$

we shall obtain a signal in the form of a sequence of rectangular pulses behind the sampler. These pulses will be distorted as a result of passing through the system, and we shall obtain at the ouput of the system the periodical waveform $x_2[n, \varepsilon]$ with the frequency of the exciting signal $x_l[n] = \mathcal{R}_e\, e^{j\bar{\omega}n}$, which will, in general, be a continuous function. Establishing the values of the parameter ε, thus assuming, for example $\varepsilon = \varepsilon_0$, we shall obtain — as a steady-state component of the signal $x_2[n, \varepsilon]$ — the function $x_{2ust}[n, \varepsilon_0]$, constituting the sinusoidal sequence

$$x_{2ust}[n, \varepsilon_0] = M_0^*(\varepsilon_0)\sin)[\omega n + \varphi^*(\varepsilon_0)]$$

.with the amplitude $M_0^*(\varepsilon_0)$ and the phase $\varphi^*(\varepsilon_0)$.

Each value of the parameter ε corresponds then (in a steady state) to a certain sequence in the form of a sinusoid, which is characterized by the amplitude $M_0^*(\varepsilon)$ and the phase $\varphi^*(\varepsilon)$. The frequency characteristic $M^*(j\bar{\omega}, \varepsilon)$ is precisely the one which determines these amplitudes and phases which are evidently the frequency functions $\bar{\omega}$ of the exciting signal $x_1[n]$.

The frequency characteristic

$$M^*(j\bar{\omega}, \varepsilon) = K_i^*(e^{j\bar{\omega}}, \varepsilon)$$

as a function of $e^{j\bar{\omega}}$, is a periodic function of frequency — that is,

$$M^*[j(\bar{\omega} + 2k\pi), \varepsilon] = M^*(j\bar{\omega}, \varepsilon); \quad k = 0, \pm 1, \pm 2, .., \tag{64}$$

This means that for two arbitrary frequencies $\bar{\omega}_1$ and $\bar{\omega}_2$ differing by the quantity $2\pi k$ (where k is an integral number), the frequency characteristic of a sampled-data system has the same value. Hence it follows that in order fully ot determine the frequency charakteristic of a sampled-data system, it is sufficient to know its plot in the interval having the length 2π — that is, for instance in the interval $-\pi \leqslant \bar{\omega} < \pi$.

A peculiarity of the frequency characteristic of a sampled-data system is its dependence on the parameter ε. Different values of ε correspond in general to different characteristics.

As a complex function, the characteristic $M^*(j\bar{\omega}, \varepsilon)$ may of course be expressed in the form of the sum of the real and the imaginary components

$$M^*(j\bar{\omega}, \varepsilon) = A^*(\bar{\omega}, \varepsilon) + jB^*(\bar{\omega}, \varepsilon) \tag{65}$$

or in the exponential form

$$M^*(j\bar{\omega}, s) = |M^*(j\bar{\omega}, \varepsilon)|e^{j\varphi^*(\bar{\omega}, \varepsilon)} \tag{66}$$

The function $|M^*(j\bar{\omega}, \varepsilon)|$ is called the amplitude characteristic, and the function $\varphi^*(\bar{\omega}, \varepsilon)$ — the phase characteristic of a sampled-data system.

From Formulae (65) and (66) follow the obvious relations

$$\begin{gathered} |M^*(j\bar{\omega}, \varepsilon)| = \sqrt{A^{*2}(\bar{\omega}, \varepsilon) + B^{*2}(\bar{\omega}, \varepsilon)} \\ \varphi^*(\bar{\omega}, \varepsilon) = \operatorname{arc\,tan} \frac{B^*(\omega, \varepsilon)}{A^*(\bar{\omega}, \varepsilon)} \end{gathered} \tag{67}$$

and

$$\begin{gathered} A^*(\bar{\omega}, \varepsilon) = |M^*(j\bar{\omega}, \varepsilon)| \cos\varphi^*(\bar{\omega}, \varepsilon) \\ B^*(\bar{\omega}, \varepsilon) = |M^*(j\bar{\omega}, \varepsilon)| \sin\varphi^*(\bar{\omega}, \varepsilon) \end{gathered} \tag{68}$$

The frequency characteristic $M^*(j\bar{\omega}, \varepsilon)$ is uniquely related with the time characteristic $b[n, \varepsilon]$. This relation may be determined, for example, on the basis of the equation of a sampled-data system

$$\mathcal{L}\{x_2[n, \varepsilon]\} = K_i^*(e^s, \varepsilon)\, \mathcal{L}\{x_1[n]\} \tag{69}$$

Substituting in Equation (69)

$$\mathcal{L}\{x_1[n]\} = \mathcal{L}\{1\} = \frac{e^s}{e^s - 1}\eta$$

and

$$b[n, \varepsilon] = x_2[n, \varepsilon]$$

we shall obtain

$$(1 - e^{-s})\, \mathcal{L}\{b[n, \varepsilon]\} = K_i^*(e^s, \varepsilon)\, \eta$$

whence

$$\frac{1}{\eta}\, \mathcal{L}\{\Delta b[n - 1, \varepsilon]\} = K_i^*(e^s, \varepsilon)$$

Thus, on account of (27) we have

$$\Delta b[n - 1, \varepsilon] = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_i^*(e^{j\omega}, \varepsilon)\, d\bar{\omega} \tag{70}$$

In terms of Formulae (30) it is also possible to set the relation between the function $b[n, \varepsilon]$ and the components $A^*(\bar{\omega}, \varepsilon)$ and $B^*(\bar{\omega}, \varepsilon)$ of the frequency characteristic; namely we have

$$\left.\begin{aligned} \Delta b[n - 1, \varepsilon] &= \frac{2}{\pi} \int_0^{\pi} A^*(\bar{\omega}, \varepsilon) \cos \bar{\omega} n d\bar{\omega}, \\ \Delta b[n - 1, \varepsilon] &= -\frac{2}{\pi} \int_0^{\pi} B^*(\bar{\omega}, \varepsilon) \sin \bar{\omega} n d\bar{\omega}. \end{aligned}\right\} \tag{70a}$$

5. CONNECTION OF SAMPLED-DATA SYSTEMS

Sampled-data systems may be connected one with another identically as "ordinary" continuous dynamical system. The following connections are thus possible: series and parallel connections, and connections in feedback systems. Series and parallel connections of sampled-data systems will be considered below.

A connection made according to the scheme indicated in Fig. 7 is called a series connection of sampled-data systems. Let us assume that the samplers of all systems connected tegether work synchronically and generate pulses having identical widths γT and the same period T.

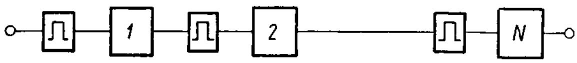


Fig. 7. Series connection of sampled-data systems

For a series connection of two sampled-data systems, we may write the following equations

$$\left.\begin{aligned} X^*_{out_1}(z,\varepsilon) &= K^*_{i_1} X^*_{in}(z), \\ X^*_{out_2}(z,\varepsilon) &= \frac{1}{k_2} K^*_{i_2}(z,\varepsilon)\, X^*_{in_2}(z), \end{aligned}\right\} \tag{71}$$

where k_2 is the amplification of the sampler in the second system.

Since at the moments of the occurrence of pulses — that is, for $\varepsilon = 0$, the signals at the primary and the secondary sides of the sampler are proportional one to another, we have then the relation

$$X^*_{in_2}(z) = k_2 X^*_{out_1}(z, 0) \tag{72}$$

Taking into consideration this relation in the second equation (71), we obtain

$$X^*_{out_2}(z,\varepsilon) = K^*_{i}(z,\varepsilon)\, X^*_{out_1}(z, 0)$$

and hence using the first equation (71) we have

$$X^*_{out_2}(z,\varepsilon) = K^*_{i_2}(z,\varepsilon)\, K^*_{i}(z, 0)\, X^*_{in}(z) \tag{73}$$

In view of the above, the pulse-transfer function of the sampled-data systems connected in series will be determined by the formula

$$K^*_{i}(z,\varepsilon) = \frac{X^*_{out_2}(z,\varepsilon)}{X^*_{in}(z)} = K^*_{i_2}(z,\varepsilon)\, K^*_{i_1}(z, 0) \tag{74}$$

By means of simple induction, Formula (74) may be generalized for the case of a series connection of N sampled-data systems, and then we obtain

$$\boxed{K^*_{i}(z,\varepsilon) = K^*_{i_N}(z,\varepsilon) \prod_{\nu=1}^{N-1} K^*_{i_\nu}(z, 0).} \tag{75}$$

It should be emphasized that this formula, in spite of a great similarity to the well-known formula determining the transfer function of a series connection of "ordinary" (that is, continuous) dynamical systems, differs from the latter very essentially. In the case of a series connection of continuous dynamical systems, the transfer function of the "series" is equal — as we know — to the product of the transfer functions of the individual systems

$$K(s) = \prod_{\nu=1}^{N} K_\nu(s) \tag{76}$$

The relation (76) is not, however, correct for sampled-data systems, since these systems in a series connection influence one another only at the moments of the occurrence of pulses (for $\varepsilon = 0$).

Therefore, the formula determining the transfer function of a series of sampled-data systems contains only the product of the quantities $K_{I\nu}(z, 0)$, that is, the transfer functions corresponding to the moments of the appearance of pulses generated by the samplers working synchronically.

Let us now consider a parallel connection of sampled-data systems. Such a connection is shown in Fig. 8. We may write the following equations

$$\left.\begin{aligned} X^*_{in_1}(z) = X^*_{in_2}(z) = \ldots = X^*_{in_N}(z), \\ X^*_{out_1}(z,\varepsilon) + X^*_{out_2}(z,\varepsilon) + \ldots + X^*_{out_N}(z,\varepsilon) = X^*_{out}(z,\varepsilon). \end{aligned}\right\} \tag{77}$$

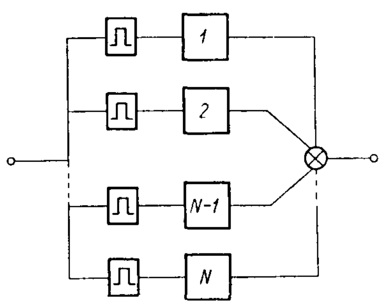


Fig. 8. Parallel connection of sampled-data systems

Moreover, for the individual sampled-data systems we have the equation

$$X^*_{out_\nu}(z,\varepsilon) = K^*_{i_\nu}(z,\varepsilon)\, X^*_{in_\nu}(z) \tag{78}$$

Hence

$$X^*_{wy}(z, \varepsilon) + [K^*_{i_1}(z, \varepsilon) + K^*_{i_2}(z, \varepsilon) + \ldots + K^*_{i_N}(z, \varepsilon)]\, X^*_{we}(z)$$

The pulse-transfer function of a parallel connection of sampled-data systems is thus the sum of the pulse-transfer functions of the individual systems

$$\boxed{K^*_i(z, \varepsilon) = \sum_{\nu=1}^{N} K^*_{i_\nu}(z, \varepsilon).} \tag{79}$$

It may be noted that Formula (79) has a form analogous to the formula determining the transfer function of a parallel connection of "ordinary" — that is, continuous — dynamical systems.

6. EXAMPLES OF INVESTIGATING SAMPLED-DATA SYSTEMS

Below we consider some examples of investigating simple sampled-data systems.

Example 1. Suppose that we consider a sampled-data system, the linear element of which is a first-order inertial element. We have to determine the function of the output signal $x_2(\bar{t}) = x_2(n, \varepsilon)$ with the assumption that at the output of the system was applied in the form of a unit-step function $x_1(t) = \mathbf{1}(\bar{t})$.

Solution. We determine the output signal $x_2(n, \varepsilon)$ using the equation of a sampled-data system

$$\mathcal{L}\{x_2[n, \varepsilon]\} = K^*_i(z, \varepsilon)\,\mathcal{L}\{x_1, [n]\}$$

Since the unit-step function $x_1(t) = \mathbf{1}(t)$ was applied at the output, then by virtue of the formula

$$\mathcal{L}\{x_1[n]\} = \mathcal{L}\{1\} = \frac{z}{z-1}\eta; \qquad z = e^s$$

we have

$$\mathcal{L}\{x_2[n, \varepsilon]\} = K^*_i(z, \varepsilon)\frac{z}{z-1}\eta \tag{80}$$

where $i = I$ for $0 \leqslant \varepsilon < \gamma$ and $i = II$ for $\gamma \leqslant \varepsilon < 1$.

On account of the assumption that the linear element of the sampled-data system is a first-order inertial element, we obtain

$$K(s) = \frac{1}{1 - sT_1}$$

and consequently (See Table 2)

$$K_I^*(z,\varepsilon) = k\left[1 - \frac{z - e^{-\beta(1-\gamma)}}{z - e^{-\beta}} e^{-\beta\varepsilon}\right]; \qquad 0 \leqslant \varepsilon < \gamma \tag{81a}$$

$$K_{II}^*(z,\varepsilon) = -k\frac{(1 - e^{\beta\gamma})z}{z - e^{-\beta}} e^{-\beta\gamma}; \qquad \gamma \leqslant \varepsilon < 1 \tag{81b}$$

where $\beta = \dfrac{T}{T_1}$.

Taking into consideration these two formulae in Equation (80), we shall obtain

$$\left.\begin{aligned} &\mathfrak{L}\{x_2[n,\varepsilon] = k\left[\frac{z}{z-1} - \frac{z - e^{-\beta(1-\gamma)}}{z - e^{-\beta}}\cdot\frac{z}{z-1}e^{-\beta\varepsilon}\right]\eta; \\ &\qquad\qquad\qquad\qquad\qquad 0 \leqslant \varepsilon < \gamma \\ &\mathfrak{L}\{x_2[n,\varepsilon]\} = -k\frac{1 - e^{\beta\gamma}}{z - e^{-\beta}}\cdot\frac{z^2}{z-1}e^{-\beta\varepsilon}\eta; \quad \gamma \leqslant \varepsilon < 1. \end{aligned}\right\} \tag{82}$$

In order to determine the function $x_2(n, \varepsilon)$, it is still necessary to calculate the inverse transformation of the expressions (82). Namely, making use of the table of the Laplace transformations of step functions (Table 1) we find

$$\mathfrak{L}^{-1}\left\{\frac{z}{z-1}\eta\right\} = \mathfrak{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

$$\mathfrak{L}^{-1}\left\{\frac{z^2}{(z - e^{-\beta})(z-1)}\delta\right\} = \frac{1}{1 - e^{-\beta}}\left(1 - e^{-\beta(n-1)}\right),$$

$$\mathfrak{L}^{-1}\left\{\frac{z}{(z - e^{-\beta})(z-1)}\delta\right\} = \frac{1}{1 - e^{-\beta}}\left(1 - e^{-\beta n}\right).$$

Accordingly

$$x_2(n,\varepsilon) = k\left[1 - \frac{1 - e^{-\beta(1-\gamma)}}{1 - e^{-\beta}}e^{-\beta\varepsilon} + \frac{1 - e^{\beta\gamma}}{1 - e^{-\beta}}e^{-\beta(n+1+\varepsilon)}\right]; \qquad 0 \leqslant \varepsilon < \gamma$$

$$x_2(n,\varepsilon) = -k\frac{e^{\beta\gamma} - 1}{1 - e^{-\beta}}\left(1 - e^{-\beta(n+1)}\right)e^{-\beta\varepsilon}; \qquad \gamma \leqslant \varepsilon < 1$$

In the course of solving the above problem, we twice used the tables — namely, in determining the pulse transfer functions $K_i^*(z, \varepsilon)$ and in determining the inverse transformation $\mathfrak{L}^{-1}$.

It is clear that tables cannot embrace all the functions encountered in practice. In this connection, in the case of the necessary results not being given in tables, the expressions $K_i^*(z, \varepsilon)$ and also the inverse transformation $\mathfrak{L}^{-1}$ should be found by means of calculations.

The calculation of the function $K_i^*(z, \varepsilon)$ for the first-order inertial element can be performed as follows.

Since the transfer function of the inertial element is expressed by the formula

$$K(s) = \frac{1}{1 + sT_1}$$

then

$$h(t) = \mathcal{L}^{-1}\left\{\frac{K(s)}{s}\right\} = 1 - e^{-\frac{t}{T_1}}$$

Substituting $t = \bar{t} \cdot T$ and $\beta = \frac{T}{T_1}$ we obtain

$$h[0, \varepsilon] = 1 - e^{-\beta\varepsilon}$$

and

$$h_\gamma[n, \varepsilon] = h[\bar{t}] - h[\bar{t} - \gamma] = (e^{\beta\gamma} - 1)\, e^{-\beta n} \cdot e^{-\beta\varepsilon}$$

Thus, in accordance with (42)

$$\begin{aligned} K_I^*(z, \varepsilon) &= k\left[1 - \frac{z - e^{-\beta(1-\gamma)}}{z - e^{-\beta}}\, e^{-\beta\varepsilon}\right]; \quad 0 \leqslant \varepsilon < \gamma, \\ K_{II}^*(z, \varepsilon) &= -k\,\frac{1 - e^{\beta\gamma}}{z - e^{-\beta}}\, ze^{-\beta\varepsilon}; \quad \gamma \leqslant \varepsilon < 1 \end{aligned} \tag{83}$$

The inverse transformation in Formulae (83) is calculated by expending rational functions into simple fractions.

Example 2. Let us now determine the function of the output signal $x_2(t)$ for the system given in Example 1, with the assumption that the input signal is $x_1(t) = \mathbf{1}(t)$ and that the width γT of generated pulses is consi-iderably smaller than the period T of the occurrence of pulses ($\gamma \ll 1$).
Solution. The function of the output signal $x_2(\bar{t}) = x_2(n, \varepsilon)$ sought for is determined in terms of the equation of a sampled-data system

$$\mathcal{D}\{x_2[n, \varepsilon]\} = K_\delta^*(z, \varepsilon)\frac{z}{z - 1}\eta \tag{84}$$

where $K_\delta^*(z, \varepsilon)$ is the pulse-transfer function corresponding to the case of very narrow pulses ($\gamma \ll 1$).

Since for $\gamma \ll 1$ we may apply the approximation

$$1 - e^{\beta\gamma} \approx -\beta\gamma$$

then, taking into consideration this approximation in Formula (81b), we find $K_\delta^*(z, \varepsilon)$

$$K_\delta^*(z, \varepsilon) = k\beta\gamma \frac{z}{z - e^{-\beta}} e^{-\beta\varepsilon}; \qquad 0 < \varepsilon < 1 \tag{85}$$

The function $K_\delta^*(z, \varepsilon)$ for the first-order inertial element is given in Table 1.

Introducing the function (85) in Equation (84), we have

$$\mathcal{L}\{x_2[n, \varepsilon]\} = k\beta \frac{z^2}{(z-1)(z - e^{-\beta})} \eta e^{-\beta\varepsilon}$$

Thus, in view of the relation

$$\mathcal{L}^{-1}\left\{\frac{z^2}{(z-1)(z-e^{-\beta})}\eta\right\} = \frac{1}{1 - e^{-\beta}}[1e^{-\beta(n+1)}]$$

we shall finally obtain

$$x_2(n, \varepsilon) = \frac{k\beta\gamma}{1 - e^{-\beta}}[1 - e^{-\beta(n+1)}]; \qquad 0 < \varepsilon < 1.$$

Table 1

Laplace transformation of step functions

No	$f[n]; n \geqslant 0$	$\frac{F^*(e^s)}{\eta} = \frac{\mathcal{L}\{f[n]\}}{\eta}$
1	1	$\frac{e^s}{e^s - 1}$
2	n	$\frac{e^s}{(e^s - 1)^2}$
3	n^2	$\frac{e^s}{(e^s - 1)^3}(e^s + 1)$
4	n^3	$\frac{e^s}{(e^s - 1)^4}(e^{2s} + 4e^s + 1)$
5	$\frac{n(n-1)}{2!}$	$\frac{e^s}{(e^s - 1)^3}$

Table 1 (continued)

No	$f[n];\ n \geqslant 0$	$\frac{F^*(e^s)}{\eta} = \frac{\mathcal{L}\{f[n]\}}{\eta}$
6	$\frac{n(n-1)(n-2)}{3!}$	$\frac{e^s}{(e^s-1)^4}$
7	$c^{\alpha n}$	$\frac{e^s}{e^s - c^\alpha}$
8	$(-c)^{\alpha n}$	$\frac{e^s}{e^s + c^\alpha}$
9	$e^{\alpha n}$	$\frac{e^s}{e^s - e^\alpha}$
10	$n e^{\alpha(n-1)};\ n \geqslant 1$	$\frac{e^s}{(e^s - e^\alpha)^2}$
11	$n^2 e^{\alpha(n-1)};\ n \geqslant 1$	$\frac{e^s}{(e^s - e^\alpha)^3}(e^s + e^\alpha)$
12	$\frac{n(n-1)}{2!} e^{\alpha(n-2)};\ n \geqslant 2$	$\frac{e^s}{(e^s - e^\alpha)^3}$
13	$\frac{1}{1-e^\alpha}(1 - e^{\alpha n})$	$\frac{e^s}{(e^s - e^\alpha)(e^s - 1)}$
14	$\frac{1}{1-e^\alpha}(1 - e^{\alpha(n+1)})$	$\frac{e^s}{(e^s - e^\alpha)(e^s - 1)} e^s$
15	$\frac{1}{1-e^\alpha}(1 - e^{\alpha(n-1)});\ n \geqslant 1$	$\frac{1}{(e^s - e^\alpha)(e^s - 1)}$
16	$\frac{1}{e^\alpha - e^\beta}(e^{\alpha n} - e^{\beta n})$	$\frac{e^s}{(e^s - e^\beta)(e^s - e^\alpha)}$
17	$\frac{n}{1-e^\alpha} - \frac{1-e^{\alpha n}}{(1-e^\alpha)^2}$	$\frac{e^s}{(e^s - e^\alpha)(e^s - 1)^2}$
18	$\cos xn$	$\frac{e^s(e^s - \cos x)}{e^{2s} - 2e^s \cos x + 1}$

No	$f[n];\ n \geqslant 0$	$\dfrac{F^*(e^s)}{\eta} = \dfrac{\mathcal{Z}\{f[n]\}}{\eta}$
19	$\sin x n$	$\dfrac{e^s \sin x}{e^{2s} - 2e^s \cos x + 1}$
20	$e^{\alpha n} \cos x n$	$\dfrac{e^s (e^s - e^\alpha \cos x)}{e^{2s} - 2 e^s e^\alpha \cos x + e^{2\alpha}}$
21	$e^{\alpha n} \sin x n$	$\dfrac{e^s e^\alpha \sin x}{e^{2s} - 2 e^s e^\alpha \cos x + e^{2\alpha}}$
22	$\cosh x n$	$\dfrac{e^s (e^s - \cosh x)}{e^{2s} - 2 e^s \cosh x + 1}$
23	$\sin x n$	$\dfrac{e^s \sinh x}{e^{2s} - 2 e^s \cosh x + 1}$
24	$e^{\alpha n} \cosh x n$	$\dfrac{e^s (e^s - e^\alpha \cosh x)}{e^{2s} - 2 e^s e^\alpha \cosh x + e^{2\alpha}}$
25	$e^{\alpha n} \sinh x n$	$\dfrac{e^s e^\alpha \sinh x}{e^{2s} - 2 e^s e^\alpha \cosh x + e^{2\alpha}}$
26	$\dfrac{\cos x n}{n\,!}$	$e^{\frac{\cos x}{e^s}} \cdot \sin\left(\dfrac{\sin x}{e^s}\right)$
27	$\dfrac{\sin x n}{n\,!}$	$e^{\frac{\cos x}{e^s}} \cdot \cos\left(\dfrac{\sin x}{e^s}\right)$
28	$\dfrac{\sin x n}{n}$	$\arctan \dfrac{\sin x}{e^s - \cos x}$
29	$\dfrac{1}{n\,!}$	$e^{\frac{1}{e^s}}$
30	$\dfrac{n+1}{n\,!}$	$\left(\dfrac{e^s + 1}{e^s}\right) e^{\frac{1}{e^s}}$
31	$\cos \pi n = (-1)^n$	$\dfrac{e^s}{e^s + 1}$
32	$n \cos \pi n = n(-1)^n$	$\dfrac{e^{2s}}{(e^s + 1)^2}$

No.	$K(s)$	$K_l^*(e^s, \varepsilon)\,;\ 0 \leqslant \varepsilon < \gamma$
1	$\dfrac{1}{ST_1+1}$	$k\left(e^{-\beta(1-\gamma)} - e^{-\beta}\right) e^{-\beta\varepsilon} \dfrac{1}{e^s - e^{-\beta}}$ $\beta = \dfrac{T}{T_1}$
2	$\dfrac{S}{ST_1+1}$	$k_1 e^{-\beta\varepsilon} \dfrac{e^s - e^{-\beta(1-\gamma)}}{e^s - e^{-\beta}}\,;$ $k_1 = \dfrac{k}{T_1}\,;\ \beta = \dfrac{T}{T_1}.$
3	$\dfrac{1}{(ST_1+1)(ST_2+1)}$	$k + \dfrac{k\beta_2}{\beta_1-\beta_2} \cdot \dfrac{e^s - e^{-\beta_1(1-\gamma)}}{e^s - e^{-\beta_1}} e^{-\beta_1\varepsilon} + \dfrac{k\beta_1}{\beta_2-\beta_1} \cdot \dfrac{e^s - e^{-\beta_2(1-\gamma)}}{e^s - e^{-\beta_2}} e^{-\beta_2\varepsilon}$ k $\beta_1 = \dfrac{T}{T_1}\,;\ \beta_2 = \dfrac{T}{T_2}.$
4	$\dfrac{1}{ST_s(ST_1+1)(ST_2+1)}$	$-k\beta_s\left(\dfrac{1}{\beta_1} + \dfrac{1}{\beta_2}\right) + k\beta_s\left(\varepsilon + \dfrac{\gamma}{e^s-1}\right) + k\dfrac{\beta_2\beta_s}{\beta_1(\beta_2-\beta_1)} \cdot \dfrac{1-e^{-\beta_1(1-\gamma)}}{e^s - e^{-\beta_1}} e^{-\beta_1\varepsilon} +$ $+ k\dfrac{\beta_1\beta_s}{\beta_2(\beta_1-\beta_2)} \cdot \dfrac{1-e^{-\beta_2(1-\gamma)}}{e^s - e^{-\beta_2}} e^{-\beta\varepsilon}\,;$ $\beta_1 = \dfrac{T}{T_1}\,;\ \beta_2 = \dfrac{T}{T_2}\,;\ \beta_s = \dfrac{T}{T_s}.$
5	$\dfrac{k_1}{ST_s(ST_1+1)}$	$-k_0\dfrac{\beta_s}{\beta} + k_0\dfrac{\beta_s}{\beta} e^{-\beta\varepsilon} \dfrac{e^s - e^{-\beta(1-\gamma)}}{e^s - e^{-\beta}} + k_0\beta_s\left(\varepsilon + \dfrac{\gamma}{e^s-1}\right);$ $k_0 = kk_1\,;\ \beta = \dfrac{T}{T_1}\,;\ \beta_s = \dfrac{T}{T_s}.$

Table 2

$K^*_{\gamma l}(e^s, \varepsilon)\,;\ \gamma \leqslant \varepsilon < 1$	$K^*_\delta\,(e^s,\varepsilon)\,;\ 0 < \varepsilon < 1$
$k\,(e^{\beta\gamma} - 1)\,e^{-\beta\varepsilon}\,\frac{e^s}{e^s - e^{-\beta}}\,;$ $\beta = \frac{T}{T_1}.$	$k\,\beta\gamma\,e^{-\beta\varepsilon}\,\frac{e^s}{e^s - e^{-\beta}}\,;$ $\beta = \frac{T}{T_1}.$
$k_1\,e^{-\beta\varepsilon}\,\frac{1 - e^{\beta\gamma}}{e^s - e^{-\beta}}\,;$ $k_1 = \frac{k}{T_1},\ \beta = \frac{T}{T_1}.$	$-\,k_1\,\beta\gamma\,e^{-\beta\varepsilon}\,\frac{1}{e^s - e^{-\beta}}\,;$ $k_1 = \frac{k}{T_1}\,;\ \beta = \frac{T}{T_1}.$
$\frac{\beta_2}{{}_1 - \beta_2}\cdot\frac{1 - e^{\beta_1\gamma}}{e^s - e^{-\beta_1}}\,e^s\,e^{-\beta_1\varepsilon} + k\,\frac{\beta_1}{\beta_1 - \beta_2}\cdot\frac{1 - e^{\beta_2\gamma}}{e^s - e^{-\beta_2}}\,e^s\,e^{-\beta_2\varepsilon}$ $\beta_1 = \frac{T}{T_1}\,;\ \beta_2 = \frac{T}{T_2}.$	$-\,k\,\frac{\beta_1\beta_2\gamma}{\beta_1 - \beta_2}\cdot\frac{e^s\,e^{-\beta_1\varepsilon}}{e^s - e^{-\beta_1}} - k\,\frac{\beta_1\beta_2\gamma}{\beta_2 - \beta_1}\cdot\frac{e^s\,e^{-\beta_2\varepsilon}}{e^s - e^{-\beta_2}}\,;$ $\beta_1 = \frac{T}{T_1}\,;\ \beta_2 = \frac{T}{T_2}.$
$k\,\beta_s\,\frac{\gamma\,e^s}{e^s - 1} + k\,\frac{\beta_2\,\beta_s}{\beta_1\,(\beta_2 - \beta_1)}\cdot\frac{(1 - e^{\beta_1})^\gamma\,e^s}{e^s - e^{-\beta_1}}\,e^{-\beta_1\varepsilon} +$ $+\,k\,\frac{\beta_1\,\beta_s}{\beta_2\,(\beta_1 - \beta_2)}\cdot\frac{(1 - e^{\beta_2})^\gamma\,e^s}{e^s - e^{-\beta_2}}\,e^{-\beta_2\varepsilon}\,;$ $\beta_1 = \frac{T}{T_1}\,;\ \beta_2 = \frac{T}{T_2}\,;\ \beta_s = \frac{T}{T_s}.$	$k\,\beta_s\,\frac{\gamma\,e^s}{e^s - 1} - k\,\gamma\,\frac{\beta_2\,\beta_s}{\beta_2 - \beta_1}\cdot\frac{e^s}{e^s - e^{-\beta_1}}\,e^{-\beta_1\varepsilon} +$ $-\,k\,\gamma\,\frac{\beta_1\,\beta_s}{\beta_1 - \beta_2}\cdot\frac{e^s}{e^s - e^{-\beta_2}}\,e^{-\beta_2\varepsilon}\,;$ $\beta_1 = \frac{T}{T_1}\,;\ \beta_2 = \frac{T}{T_2}\,;\ \beta_s = \frac{T}{T_s}.$
$k_0\,\beta_s\,\frac{\gamma\,e^s}{e^s - 1} - k_0\,\frac{\beta_s}{\beta}\,(e^{\gamma\beta} - 1)\,e^{-\beta\varepsilon}\,\frac{e^s}{e^s - e^{-\beta}}\,;$ $k_0 = kk_1\,;\ \beta = \frac{T}{T_1}\,;\ \beta_s = \frac{T}{T_s}.$	$k_0\,\beta_s\,\frac{\gamma\varepsilon^s}{e^s - 1} - k_0\,\beta_s\,\gamma\,e^{-\beta\varepsilon}\,\frac{e^s}{e^s - e^{\beta}}\,;$ $k_0 = kk_1\,;\ \beta = \frac{T}{T_1}\,;\ \beta_s = \frac{T}{T_s}.$

PART III

OUTLINE OF THE THEORY AND PRINCIPLES OF DESIGNING SAMPLED-DATA CONTROL SYSTEMS

1. REVIEW OF MORE IMPORTANT TYPES OF SAMPLED-DATA CONTROL SYSTEMS

Sampled-data control systems are at present often used in different branches of control engineering. They are particularly often applied in the problems of control of slowly changing processes — for example, pressure and temperature in boilers, temperature in industrial furnaces, etc., because in addition to the advantage of simplifying the equipment, sampled-data control systems make it possible to attain a more profitable control process. In addition, these systems find a broad application in radiolocation, remote measurements and remote control.

A block diagram of a sampled-data control system is shown in Fig. 9.

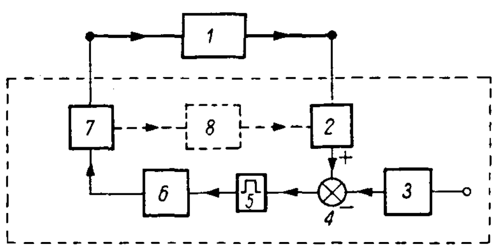


Fig. 9. Block diagrams of a sampled-data control system (1) controlled system, (2) measuring element, (3) input element, (4) summing element, (5) sampler, (6) correcting element, (7) element of the auxiliary feedback

A typical property of sampled-data control is the presence in the feedback branch of a sampler transforming the signal usually constituting a continuous function into a signal in the form of a sequence of rectangular pulses.

We distinguish three main types of sampled-data control systems depending on the construction of the sampler.

To the first type belong systems with samplers transforming the input signal into a signal in the form of rectangular pulses with **heights** proportional to the values of the input signal, at the moments of occurrence of pulses. The period T and the width γT of pulses generated by samplers are constant quantities for such systems. The principle of operation of such a sampler has already been discussed; an ideal scheme of it is shown in Fig. 4.

To the second type of sampled-data control systems belong those with samplers transforming the input signal into a signal in the form of rectangular pulses the **widths** of which are proportional to the values of the input signal at the moment of the appearance of pulses. The heights of pulses in systems of this kind are constant, and thus do not depend on the input signal. The operation principle of the sampler in the second type of systems is explained in Fig. 10. The bar *1* set into periodic motion by means of a special mechanism *5* presses the indicating needle *2* to the conducting surface *3* which is split and insulated in the cross section "$a - a$". Two batteries *4* with electromotoric forces E_O are connected to the conducting surfaces. Similarly as in the case of samplers of the first type, the input signal $x_1(t)$ of the sampler is the deflection of the indicating needle *2* from the central position. The output signal $x_2(t)$ is the voltage on the terminals a, b. The signal has the form of rectangular pulses with constant heights, but — owing to the wedge-shaped bar *1* — differing one from another as to their width and sign, which are dependent on the input signal $x_1(t)$ (Fig. 11).

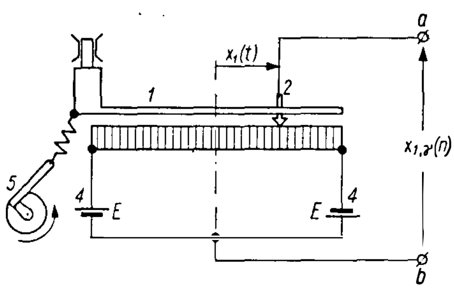


Fig. 10. Operation principle of the second-type mechanical sampler (1) bar, (2) indicating needle of the galvanometer, (3) conducting surface, (4) batteries of voltage E_0, (5) driving mechanism

Fig. 11. Pulse signal generated by the second-type sampler

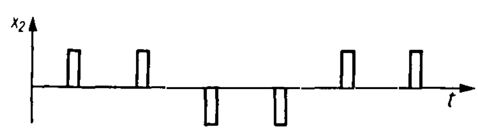


Fig. 12. Pulse signal generated by the third-type sampler

To the third type of sampled-data control systems belong those with samplers transforming the input signal into a signal in the form of rectangular pulses having constant heights and widths but differing one from another by their **sign**, which is dependent on the input signal (Fig. 12).

The operation principle of this type of samplers is explained in Fig. 13. Such a sampler differs from the one previously discussed only by the shape of the bar which in the present case is rectangular.

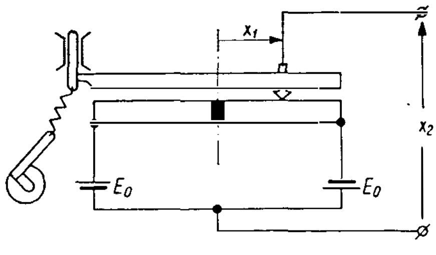


Fig. 13. Operation principle of the third-type sampler

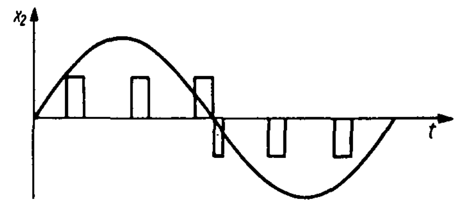


Fig. 14. Pulse signal in the case of a delay system with a key

Control systems of the first and the second type may be considered as belonging to the category of linear systems, because in the case of those systems certain parameters (the height or the width of pulses) of the signal correcting the controlled quantity are proportional to the value of the controlling signal. Systems of the third type are classified with nonlinear systems. There is a great similarity between this type of systems and systems with relay control; namely, if a system with relay control is provided with a breaking key, then such a system will differ as regards the manner of working from a sampled-data control system of the third type only in that during the duration of a pulse there may occur in it a change of the pulse sign (as a result of the change of the sign in the input signal) (Fig. 14).

In our further considerations, we shall confine ourselves to the investigation of sampled-data control systems only — that is, to the systems of the first and second type.

Note that systems of the first type are most often applied in radiolocation engineering, in remote measurements; however, in industrial control engineering — namely in the control of slowly changing processes, such as temperature, concentration, pressure-systems of the second type are mostly applied, that is, systems with varying widths of pulses.

An important advantage of sampled-data control systems of the second type is the full utilization of the power of the operating element — for instance of the servomotor which operates with a constant angular velocity. In systems of the first type, full utilization of the power of a servomotor may take place only at the moment of the highest values of the controlling signal. For the rotational speed of the servomotor (or another corresponding quantity) is proportional to the height of rectangular pulses generated by the sampler.

Another advantage of a sampled-data control system of the second type is that the sampler of such a system removes the effect of non-linearity of a nonlinear element. This happens when the nonlinear element on which the sampler acts has a symmetric characteristic (Fig. 15).

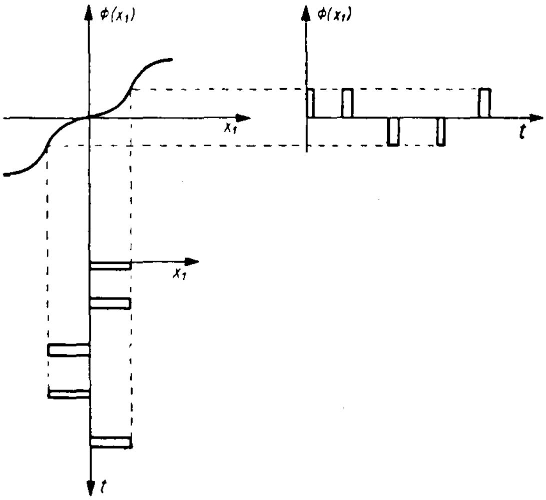


Fig. 15. Explanation of the elimination of the second-type non-linearity effect by a sampler

In the Part II of this paper, we were concerned with the method of investigating sampled-data systems of the first type. Now we shall prove that, with suitable assumptions, this method can also be applied to investigating sampled-data systems of the second type — that is, those provided with samplers generating pulses with varying widths. Namely, let us pursue the following reasoning.

It should be noted that for a sampled-data system of the second type, the signal at the output of the sampler is a sequence of rectangular pulses with constant heights, which are equal to

$$k \operatorname{sign} x_1(n)$$

and with the widths $\gamma(n)$ proportional to the absolute values of the function $x_1(n)$

$$\gamma(n) = \varkappa |x_1(n)| \tag{86}$$

Thus if we denote the response of the dynamical element to a unit-step excitation by the symbol $h(\bar{t})$, then in the interval $n + \gamma \leqslant \bar{t} < n + 1$ the output signal $x_2(\bar{t})$ of the sampled-data system of the second type will be determined by the formula

$$x_2(\bar{t}) = \sum_{m=0}^{\infty} k \operatorname{sign} x_1(n)[h(\bar{t} - m) - h(\bar{t} - m - \gamma)]$$

Taking then the assumption

$$\max \gamma(n) \ll 1 \tag{87}$$

which is satisfied in the case of small values of the input variable $x_1(n)$, we may use the approximation

$$h(\bar{t} - m - \gamma) \approx h(\bar{t} - m) - \gamma h'(\bar{t} - m) \tag{88}$$

Hence

$$x_1(\bar{t}) \approx \sum_{m=0}^{\infty} k \operatorname{sign} x_1(m) \gamma(m) h'(\bar{t} - m) =$$

$$= k\varkappa \sum_{m=0}^{\infty} |x_1(m)| \operatorname{sign} x_1(m) h'(\bar{t} - m)$$

and owing to the obvious equality

$$|x_1(m)| \operatorname{sign} x_1(m) = x_1(m)$$

$$x_2(\bar{t}) \approx l \sum_{m=0}^{\infty} x_1(m) h'(\bar{t} - m); \qquad n < \bar{t} <_n + 1$$

where

$$l = k\varkappa.$$

Thus, after expressing the above formula by means of step functions dependent on the parameter ε, and after applying the Laplace transformation, we shall obtain

$$\mathcal{L}\{x_2[\bar{t}, \varepsilon]\} \approx K_\gamma^*(z, \varepsilon) \mathcal{L}\{x_1[\bar{t}]\}; \quad 0 < \varepsilon < 1 \tag{89}$$

where

$$K_\gamma^*(z, \varepsilon) = \frac{\mathcal{L}\{k\varkappa h'[\bar{t}, \varepsilon]\}}{\eta}$$

Therefore, owing to Approximation (88), the properties of a sampled--data system of the second type can be expressed by means of the function

$$K_\gamma^*(z, \varepsilon) = K_\gamma^*(e^s, \varepsilon) = \frac{\mathcal{L}\{k\varkappa h'[\bar{t}, \varepsilon]\}}{\eta}$$

which, as before, we call the pulse transfer function of sampled-data systems of the second type.

The functions $K_\gamma^*(z, \varepsilon)$ for more important dynamical systems are appended in Table 2.

In view of Formula (89), in considering sampled-data systems of the second type, we can fully utilize the method presented in Part II of this paper. Concluding, we should note that Approximation (88) which conditions the application of this method is practically admissible provided that the sampled-data system under consideration is characterized by a sufficient sensitivity.

2. EQUATIONS OF A SAMPLED-DATA CONTROL SYSTEM

Every control system, and thus in particular also a sampled-data control system is — as regards its structure — a system with a negative feedback. In a control system, we can always distinguish two basic elements — namely, a controlled system and a controller. A controller of the simplest sampled-data control system consists of a measurement element, a sampler and an operating element — for example a servomotor (Fig. 16).

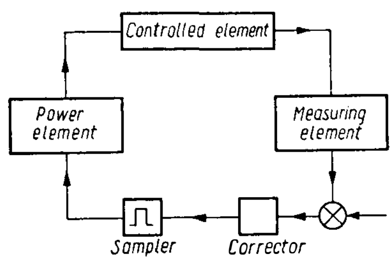


Fig. 16. Simple sampled-data control system

If we denote the pulse transfer function of a system of the first type, which is open before the sampler (that is at the point "a") by the symbol $K_i^*(z, \varepsilon)$, then for a closed system the following formula will hold

$$X_{wy}^*(z, \varepsilon) = \frac{K_i^*(z, \varepsilon)}{1 + K_i^*(z, 0)} \, X_0^*(z, 0). \tag{90}$$

Formula (90) is called the "equation of a sampled-data feedback system", and the expression

$$K_{s,i}^*(z, \varepsilon) = \frac{K_i^*(z, \varepsilon)}{1 + K_i^*(z, 0)} \tag{91}$$

is called the transfer function or the characteristic function of a sampled--data feedback system.

In the case of very narrow pulses, evidently we have to substitute the functions $K_\delta^*(z, \varepsilon)$ and $K_\delta^*(z, 0)$ for $K_I^*(z, \varepsilon)$ and $K_I^*(z, 0)$ in Formula (91); then Formula (91) will take the form

$$K_{s,\delta}^*(z, \varepsilon) = \frac{K_\delta^*(z, \varepsilon)}{1 + K_\delta^*(z, 0)} \tag{91a}$$

For sampled-data systems of the second type, we substitute the functions $K_\gamma^*(z, \varepsilon)$ and $K_\gamma^*(z, 0)$ for $K_i^*(z, \varepsilon)$ and $K_I^*(z, 0)$; hence we have

$$K_{s,\gamma}^*(z, \varepsilon) = \frac{K_\gamma^*(z, \varepsilon)}{1 + K_\gamma^*(z, 0)} \tag{91b}$$

Formula (91) (and also Formulae (91a) and (91b)) differs from the analogous formula determining the transfer function of a system with an "ordinary" (that is, continuous) feedback, primarily in that it contains in the denominator the function $K^*(z, 0)$ corresponding to the moments $\varepsilon = 0$. This fact is an obvious consequence of the presence of the sampler in the feedback branch. Owing to the sampler, the feedback branch is closed only at the moments $\varepsilon = 0$, and therefore the function $K_I^*(z,$ appears in Formula (91) instead of the function $K^*(z, \varepsilon)$, and the functions $K_\delta^*(z, 0)$ or $K_\gamma^*(z, 0)$ in Formulae (91a) and (91b).

3. STABILITY OF A SAMPLED-DATA CONTROL SYSTEM

3.1. Frequency criteria of stability

A basic condition of correct operation of control systems — and thus in particular of sampled-data control systems — is their stability. As we know, by the stability of a dynamical system, we mean the property consisting in that its pulse response (the output signal actuated by an excitation by means of the Dirac function) is a decaying function, and accordingly it satisfies the condition

$$\lim_{t \to \infty} k(t) = 0$$

From the theory of dynamical systems, we know that for a continuous-data linear system to be stable, it is necessary and sufficient that the poles s_ν of the transfer function $K(s)$ of the system have negative real parts

$$\mathcal{Re}\, s_\nu < 0$$

that is, that the poles s_ν lie in the left semiplane of the complex variable s.

It is clear that each sampled-data system containing no feedback is stable when its linear element is stable. The investigation of stability of sampled-data systems containing no feedbacks reduces to the problem of investigating its linear part, and thus offers no major difficulties. However, the investigation of stability of sampled-data feedback systems requires the application of special methods.

Let us consider the transfer function of a sampled-data feedback system

$$K_{s,i}^*(e^s, \varepsilon) = \frac{P_i^*(e^s, \varepsilon)}{Q^*(e^s)} = \frac{K_i^*(e^s, \varepsilon)}{1 + K_i^*(e^s, 0)}$$

It is evident that the sampled-data system is stable if and only if the poles of the function $K_{s,i}^*(e^s, \varepsilon)$ lie in the semiplane $\mathcal{R}_e s < 0$ for the rational function $K_{s,i}^*(e^s, \varepsilon) = \frac{\mathcal{L}\{x_2[\bar{t}, \varepsilon]\}}{\mathcal{L}\{x_0[t]\}}$ can always be expanded into simple fractions of the form

$$\frac{e^s}{(e^s - e^{s_\nu})^\mu}$$

which as Laplace images of step functions represent the following functions $f_\nu[n]$

$$f_\nu[n] = \mathcal{L}^{-1}\left\{\frac{e^s}{(e^s - e^{s_\nu})^\mu}\ \eta\right\} = \frac{n(n-1)\ldots(n-\mu+2)}{(\mu-1)!}\ e^{s_\nu(n-\mu+1)}$$

However, the functions $f_\nu(n)$ satisfy the condition

$$\lim_{n\to\infty} f_\nu[n] = 0$$

if and only if $\mathcal{R}_e\, s_\nu < 0$ that is, only when the poles of the transfer function $K_{s,i}^*(e^s, \varepsilon)$ lie in the left semiplane of the complex variable s.

A direct calculation of the poles s_ν for investigating whether they satisfy the condition $\mathcal{R}_e\, s_\nu < 0$, is very troublesome, and in a general case (with a high degree of the polynomial $Q^*(e^s)$) it is possible only in an approximate manner. Therefore, in practice, in order to determine the stability of a sampled-data control system, we use methods requiring no knowledge of the poles of the function $K_{s,i}^*(e^s, \varepsilon)$. One of such is the method based on frequency characteristic, and constituting an adaptation of the Nyquist stability criterion in the theory of sampled-data systems. This method is presented here below.

Let us consider the function

$$\varphi^*(e^s) = 1 + K_i^*(e^s, 0) \tag{92}$$

constituting the denominator of the transfer function $K_{s,i}^*(e^s, \varepsilon)$ with a sampled-data feedback system.

The zeroes of the function $\varphi^*(e^s)$ are evidently zeroes of the transfer function $K^*_{s,l}(e^s, \varepsilon)$ and the poles of the function $\varphi^*(e^s)$ are poles of $K^*(e^s, 0)$. Since $K^*_l(e^s, 0)$ is the transfer function of an "open" sampled-data system, hence it follows that in the case in which an open system is stable, the poles of the function $\varphi^*(e^s)$ lie in the left semiplane of the complex variable s.

Let us choose on the plane s the contour C, shaped as a rectangle and plotted as shown in Fig. 17, and assume that the abscissa of the segment C_4 of the contour C increases to infinity

$$\sigma_0 \to \infty$$

If we assume that on the contour C the function $\varphi^*(e^s)$ has no zeroes and no poles, then by virtue of the theorem concerning the increase in the argument, we shall obtain the following formula for a single passage over the contour C in the clock-wise direction

$$\Delta_c \arg \varphi^*(e^s) = 2\pi(M - N) \tag{93}$$

where M is the number of poles, and N the number of zeroes of the function $\varphi^*(e^s)$ lying in the region limited by the contour C.

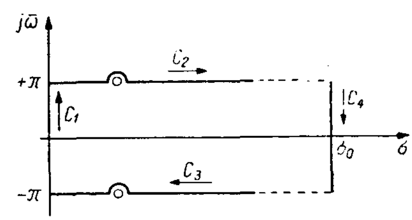


Fig. 17. C-contour by means of which the stability of a sampled--data control system is determined

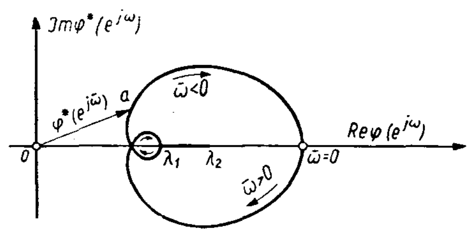


Fig. 18. Plot of the function $\varphi^*(e^s)$

If the system under consideration is stable with the open feedback branch, then — in accordance with the above remark — $M = 0$, and

$$\Delta_c \arg \varphi^*(e^s) = -2\pi N$$

Notice that in the case of stating that $N = 0$, owing to the periodicity of the function $\varphi^*(e^s)$, this function has no zeroes not only in the region under consideration, but also in the whole semiplane $\mathcal{R}_e\, s \geqslant 0$. Hence the condition of stability of the sampled-data feedback system is expressed by the formula

$$\Delta_c \arg \varphi^*(e^s) = 0 \tag{94}$$

Since the obvious equality

$$\varphi^*(e^{-j\pi}) = \varphi^*(e^{j\pi})$$

holds, then with the change of the veriable s along the imaginary axis $j\omega$ from $-j\pi$ to $j\pi$, the function $\varphi^*(e^s)$ will correspond to a certain closed curve

$$\varphi^*(e^{j\bar{\omega}}) = 1 + K_I^*(e^{j\bar{\omega}}, 0)$$

This curve, drawn on the plane with the rectangular coordinates $\mathcal{R}_e\,\varphi^*(e^{j\bar{\omega}})$, $\mathcal{I}_m\,\varphi^*(e^{j\bar{\omega}})$ intersects the real axis at least twice — namely at the points $\bar{\omega} = 0$ and $\bar{\omega} = \pm\pi$. The above property follows immediately from the following equalities

$$e^{j0} = 1 \text{ i } e^{\pm j\pi} = -1$$

consequent upon which the expressions

$$K_I^*(e^{j0}, 0) \text{ and } K_I^*(e^{j\pi}, 0)$$

are at these points real numbers.

With the change of the variable s along the straight line C_2 from $\sigma = 0$ to $\sigma = \infty$, the function $\varphi^*(e^s)$ will constantly remain a real quantity (Fig. 18) — that is, will vary along the real axis, for instance

$$\text{from } \varphi^*(e^{j\pi}) = \lambda_1 \text{ to } \lim_{\sigma\to\infty} \varphi^*(e^{\sigma+j\pi}) = \lambda_2$$

Subsequently, with the change of s along the straight line C_2 and with the assumption that $\sigma \to \infty$, the function $\varphi^*(e^s)$ will not change its value. With the change of the variable s along the straight line C_3 from $\sigma = \infty$ to $\sigma = 0$, the function $\varphi^*(e^s)$ will again vary along the real axis from λ_2 to λ_1, since we have

$$\varphi^*(e^{\sigma-j\pi}) = \varphi^*(e^{\sigma+j\pi})$$

Summarizing what has been said we state that the increment of the argument of the function $\varphi^*(e^s)$ along a closed contour C is entirely determined by the curve corresponding to the change of the variable s along the imaginary axis from $\bar{\omega}_1 = -\pi$ to $\bar{\omega}_2 = \pi$, since we have

$$\Delta_c \arg \varphi^*(e^s) = \underset{\bar{\omega}\varepsilon[-\pi,\pi]}{\Delta} \arg \varphi^*(e^{j\bar{\omega}}) \tag{95}$$

Notice that the total increase in the argument of the function $\varphi^*(e^s)$ along the contour C is determined by the number of rotations of the vector $\vec{Oa} = \varphi^*(e^{j\bar{\omega}})$ with change in frequency from $-\pi$ to π (Fig. 18). The increase in the argument of the function $\varphi^*(e^s)$ along the contour C is then equal to zero only in the case in which the plot of the curve $\varphi^*(e^{j\bar{\omega}})$ does not comprise the origin of the coordinate system.

Let us now consider the function

$$K_I^*(e^{j\bar{\omega}}, 0) = |K_I^*(e^{j\bar{\omega}}, 0)|\, e^{j\Theta(\bar{\omega}, 0)}$$

which represents the frequency characteristic (for $\varepsilon = 0$) of an open sampled-data feedback system. The plot of the function $K_I^*(e^{j\bar{\omega}}, 0)$ is obtained from the diagram of $\varphi^*(e^{j\bar{\omega}})$ by shifting the origin of the coordinate system "to the right" by the quantity 1 (Fig. 19). The function $\varphi^*(e^{j\bar{\omega}})$ in Fig. 19 is then represented by the vector having origin at the point $-1, j0$ and accordingly the stability of a sampled-data feedback system is expressed by the following theorem.

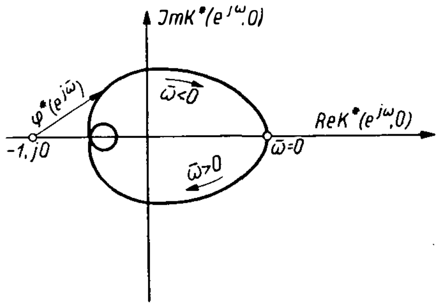


Fig. 19. Nyquist diagram of an open sampled-data control system

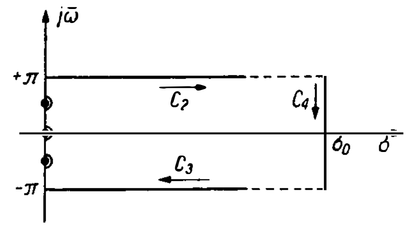


Fig. 20. C — contour in the case of poles lying on the imaginary axis

Theorem 1. For a sampled-data feedback system to be a stable system, in is necessary and sufficient that the Nyquist diagram $K^*(e^{j\bar{\omega}}, 0)$ of a stable open sampled-data system shall not contain the point -1, j0 for the frequency change from $-\pi$ to π.

On account of the symmetry (with respect to the real axis) of the characteristic $K_l^*(e^{j\bar{\omega}}, 0)$ for $0 \leqslant \bar{\omega} < \pi$ and $-\pi \leqslant \bar{\omega} < 0$ in determining the Nyquist diagram we may, in the interests of simplicity, disregard the branch of the curve corresponding to $0 \leqslant \bar{\omega} < \pi$.

In proving Theorem 1, we assumed that a system is stable after the feedback branch has been opened. In practice, however, we often use a control system which does not satisfy this condition. Of such are, for example, the "astatic systems" — that is, systems containing integrating elements. It is then necessary to extend the frequency criterion of stability so that they shall cover the case of astatic systems. In this connection, we shall follow the reasoning given below.

Let us assume that the function $\varphi^*(e^{j\bar{\omega}})$ has poles on the imaginary axis. At the poles of the function $\varphi^*(e^{j\bar{\omega}})$ the characteristic $K_l^*(e^{j\bar{\omega}}, 0)$ takes, of course, an infinite value. Considering a closed contour C' on the complex variable plane s (Fig. 20), which differs from the contour C from Fig. 17 in that by semicircles having infinitely small radia, it sweeps round the poles lying on the imaginary axis, we shall obtain, for systems having the characteristic under consideration — by virtue of the theorem concerning the increase of the argument — a stability analogous to that given above. In the case under consideration, with the change of s along "infinitely small" semicircles, the vector $\varphi^*(e^{j\bar{\omega}})$, and thus also $K_I^*(e^{j\bar{\omega}}, 0)$ will vary along arcs having radia tending to infinity and the angle equal to πr, where r is the order of the pole.

For first-order astatic systems which are most often applied in practice, the function $\varphi^*(e^s)$ has a first-order pole at the origin of the coordinate system. In investigating the stability of such systems, we may then use the theorem given above, provided that the Nyquist diagram $K_I^*(e^{j\bar{\omega}}, 0)$ of an "open" system is supplemented by an arc having the radius $R \to \infty$. The orientation of this arc must be opposite to the direction corresponding to that of sweeping round the pole; it must then have a clock-wise direction.

The characteristics of stable (a) and an unstable (b) astatic sampled--data control systems are shown in Fig. 21.

Using the frequency criterion of stability, we can determine the limit values of the parameters of a system — for example, the greatest total amplification in the feedback loop, or the greatest admissible width of pulses, for which the system is at the limit of stability. This problem is illustrated by the following example.

Example. The first pulse-transfer function of a system of the first type with an open feedback branch is given by the formula:

$$K_I^*(z, \varepsilon) = k\left[1 - \frac{z - e^{-\beta(1-\gamma)}}{z - e^{-\beta}} e^{-\beta\varepsilon}\right]; \qquad 0 \leqslant \varepsilon < \gamma. \tag{96}$$

Determine the greatest admissible amplification k which ensures a stable operation of the system after the feedback branch has been closed.

Solution. The stability of a sampled-data feedback system is dependent on the frequency characteristic $K_I^*(e^{j\bar{\omega}}, 0)$. According to (96), we have

$$K_I^*(e^{j\bar{\omega}}, 0) = k\left[1 - \frac{e^{j\bar{\omega}} - e^{-\beta(1-\gamma)}}{e^{j\bar{\omega}} - e^{-\beta}}\right]$$

The diagram of this characteristic for the interval $0 \leqslant \omega < \pi$ — drawn on the plane with the coordinates $\mathcal{R}_e\, K_I^*(n^{j\bar{\omega}}, 0)$, $\mathcal{I}_m\, K_I^*(e^{j\bar{\omega}}, 0)$ — is shown in Fig. 22.

By virtue of Theorem 1, a system with a closed feedback loop is stable, when the Nyquist diagram does not contain the point $-1, j0$ — that is, when the inequality

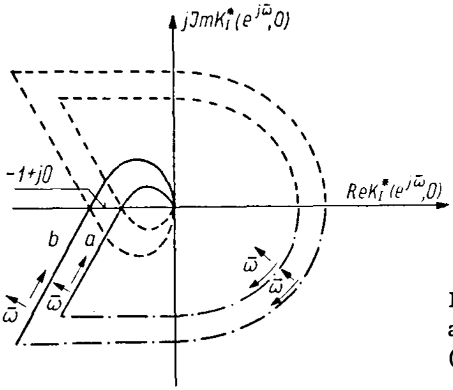
$$K_I^*(e^{j\pi}, 0) > -1$$

is satisfied. However, since in view of Formula (96) we have

$$K_I^*(e^{j\pi}, 0) = -k(e^{\beta\gamma} - 1)\frac{e^{-\beta}}{1 + e^{-\beta}}$$

then the system will be stable for

$$k < k_{lim} = \frac{1 + e^{-\beta}}{e^{-\beta(1-\gamma)} - e^{-\beta}} \tag{97}$$



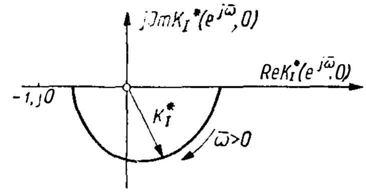


Fig. 22. Nyquist diagram of a simple sampled-data system

Fig. 21. Nyquist diagrams (a) of a stable astatic sampled-data control system, (b) of an unstable astatic sampled-data control system

Using the relation (97), the parameters β and γ being given, we can determine the greatest admissible amplification k in the feedback loop.

3.2. Time criteria of stability

3.2.1. Introduction

The methods of investigating the stability of sampled-data control systems, discussed in the preceding chapter, are based on the frequency characteristics of sampled-data systems. These methods, valuable as they are in designing, have a certain practical inconvenience. Namely, since experimental determination of the frequency characteristics of sampled-data systems is almost impossible, then these methods are practically useless in the cases in which the analysis of a system must be based on the data obtained from measurements.

Below is given a method making it possible to determine the stability of a sampled-data system directly from the knowledge of the time characteristic of an "open system". Moreover, an analysis is effected concerning typical time characteristics, which yields diagrams determining the "stability zones" of sampled-data control systems under consideration. The results obtained make it possible in certain cases to investigate stability almost immediately from the knowledge of the time characteristic (the response to a unit-step excitation) of an open control system, which can relatively simply be determined experimentally. Note that such a procedure is applied by certain authors in the case of continuous-data control systems.

We find it convenient as regards the method of analysis performed, to distinguish the static and astatic systems. The use of such a classification of control systems is of course arbitrary.

The methods of investigating stability, as presented in this chapter, were earlier presented by the present author in the paper entitled "The numerical operator method". Below is given a slightly different treatment of the same results.

3.2.2. Determination of the stability of a sampled-data system from knowledge of the open system time characteristic

Let us consider a sampled-data feedback system (Fig. 23). If the feedback loop is broken before the sampler (at the point a), then we obtain an ordinary sampled-data system, the theory of which was given in Part II.

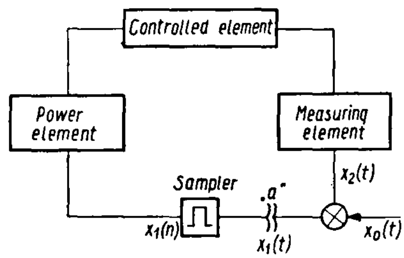


Fig. 23. Block diagram of a sampled-data control system

Let us write the equation of an open system at the point (a)

$$\mathcal{L}\{x_2(n,\varepsilon)\} = K_i^*(z,\varepsilon)\,\mathcal{L}\{x_1(n)\} \tag{98}$$

Assume now that at its output — that is at the point (a) — an excitation is appled in the from of a unit-step function. At the output of the sampler, there will then appear a sequence of rectangular pulses with identical unit heights; thus we shall obtain

$$\mathcal{L}\{x_1(n)\} = \mathcal{L}\{1\} = \frac{z}{z-1}\,\eta \tag{99}$$

hence the signal $x_2(t)$ may be expressed by the formula

$$\mathcal{L}\{x_2(n,\varepsilon)\} = K_i^*(p,\varepsilon)\frac{z}{z-1}\,\eta \tag{100}$$

If, in turn, we assume that the (open) sampled-data system under consideration is stable, consequent upon which

$$\lim_{n\to\infty}[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)] = 0^* \tag{106}$$

then in formula (102) it will be possible to drop all the terms with sufficiently high coefficients ($n > N$), for which

$$|x_2(n,\varepsilon) - x_2(n-1,\varepsilon)| < \delta \tag{107}$$

where δ we may practically take, for example, 0.1 k or 0.05 k, where k is the total amplification in the feedback loop. We shall then obtain the approximate formula

$$K_i^*(e^{j\bar{\omega}},\varepsilon) \approx x_2(0,\varepsilon) + \sum_{n=1}^{N}[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)]\, e^{-j\bar{\omega}n} \tag{108}$$

which is sufficiently accurate for determining the stability of a closed sampled-data control system.

According to the results obtained in the preceding chapter, we determine the stability of a system from the following

Theorem. A sampled-data feddback system is stable, if and only if the Nyquist diagram $K^*(e^{j\bar{\omega}}, 0)$ of the open system does not contain the point -1, $j0$, with the change of $\bar{\omega}$ from $-\pi$ to $+\pi$.

In view of this theorem, the method of investigating the stability of a sampled-data control system in terms of the time characteristic $x_2(n, \varepsilon)$ of the open system, is reduced to:

1° determining on the basis of Formula (108) the function $K^*(e^{j-}, 0)$, from the knowledge of the time characteristic $x_2(n, \varepsilon)$ of the open system;

2° drawing the Nyquist diagram; the stability of the system is then determined on the basis of the theorem cited above.

The Nyquist diagram $K^*(e^{j\bar{\omega}}, 0)$ can be found in the easiest way by calculation of its components

$$\left.\begin{aligned} \mathcal{R}_m K^*(e^{j-},0) &= x_2(0,0) + \sum_{n=1}^{N}[x_2(n,0) - x_2(n-1,0)]\cos n\bar{\omega}, \\ \mathcal{I}_m K^*(e^{j\bar{\omega}},0) &= -\sum_{n=1}^{N}[x_2(n,0) - x_2(n-1,0)]\sin n\bar{\omega}. \end{aligned}\right\} \tag{109}$$

From the considerations presented above, we may draw an interesting conclusion — namely, that the stability of a sampled-data control system depends only and exclusively on the plot of the time charac-

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* This formula is a necessary but not sufficient condition of the stability of a sampled-data system.

Since the image of a step function can be represented in the form of. the series power

$$\mathcal{L}\{f(\bar{t})\} = \sum_{n=0}^{\infty} f(n) z^{-n} \eta \tag{101}$$

then we shall obtain

$$K_i^*(z, \varepsilon) = \frac{z-1}{z} \sum_{n=0}^{\infty} x_2(n, \varepsilon) z^{-n} = (1 - z^{-1}) \sum_{n=0}^{\infty} x_2(n, \varepsilon) z^{-n} =$$

$$= x_2(0, \varepsilon) + \sum_{n=0}^{\infty} [x_2(n, \varepsilon) - x_2(n-1, \varepsilon)] z^{-n}. \tag{102}$$

From Equation (102), we can determine the frequency characteristic of a sampled-data system. Namely, in accordance with the former considerations if the frequency characteristic $M^*(j\bar{\omega}, \varepsilon)$ of a sampled-data system is determined by the formula

$$M^*(j\bar{\omega}, \varepsilon) = \frac{x_{2steady}(n, \varepsilon)}{x_1(n)} = \frac{x_{2steady}(n, \varepsilon)}{ce^{j\omega n}} \tag{103}$$

where $x_{2\,steady}(n, \varepsilon)$ is the steady-state component of the output signal $x_2(n, \varepsilon)$, and

$$x_1(n) = ce^{j\bar{\omega}n} \tag{104}$$

is the exciting signal, then we can easily prove the following.

Property. The frequency characteristic $M^*(j\bar{\omega}, \varepsilon)$ of a sampled-data system is equal to the expression $K_i^*(e^{j\bar{\omega}}, \varepsilon)$ which is obtained by substitution, in the transfer function $K_i^*(z, \varepsilon)$, the displacement operator z for the function $e^{j\bar{\omega}}$:

$$M^*(j\bar{\omega}, \varepsilon) = K_i^*(e^{j\bar{\omega}}, \varepsilon) \tag{105}$$

The above property can be proved in terms of the equation of a sampled-data system, with the assumption that $x_1(n) = c\, e^{j\bar{\omega}n}$. After elementary transformation, we arrive at Formula (103).

By virtue of Formula (103), the frequency characteristic can then be determined from knowledge of $x_2(n, \varepsilon)$

$$M^*(j\bar{\omega}, \varepsilon) = K_i^*(e^{j\bar{\omega}}, \varepsilon) = x_2(0, \varepsilon) + \sum_{n=1}^{\infty} [x_2(n, \varepsilon) - x_2(n-1, \varepsilon)] e^{-j\bar{\omega}n}$$

where $x_2(n, \varepsilon)$ is the function of the output signal actuated by a unit-step excitation

$$x_1(t) = \mathbf{1}(t)$$

teristic $x_2(\bar{t})$ at the moments $\bar{t} = \frac{t}{T} = n$ $(n = 0, 1, 2, \ldots)$ — that is, on the plot $x_2(n, 0)$. The behavior of the characteristic $x_2(\bar{t})$ between the moments $\bar{t} = n$ has absolutely no bearing on the stability of the system. This fact becomes obvious if we notice that owing to the presence of the sampler in the feedback loop, the control system is closed only at the moments $\bar{t} = n$.

The plot of the time charakteristic $x_2(n, \varepsilon)$ can evidently be determined experimentally or can be calculated by the analytical method. It should be emphasized that the method presented is suitable for the investigation of systems which are stable after the opening of the feedback branch — that is, to the investigation of systems which do not contain, for example, elements with astatic characteristics.

Below, it will be proved that this restriction may be avoided in a relatively simple way. Namely, notice that for an open sampled-data system having an astatic characteristic and containing only one integrating element, the following relation must always be satisfied

$$\lim_{n \to \infty} \Delta^2 x_2(n - 1, \varepsilon) = 0; \qquad 0 \leqslant \varepsilon < 1 \tag{110}$$

where

$$\begin{aligned} \Delta^2 x_2(n - 1, \varepsilon) &= \Delta [x_2(n, \varepsilon) - x_2(n - 1, \varepsilon)] = \\ &= x_2(n + 1, \varepsilon) - 2x_2(n, \varepsilon) + x_2(n - 1, \varepsilon), \end{aligned} \tag{111}$$

and $x_2(n, \varepsilon)$ is the response of the sampled-data system to a unit-step excitation.

Thus, if the formula determining $K_i^*(e^{j\bar{\omega}}, 0)$ is transformed so that under the sign of the sum there appear terms of the form $\Delta^2 x_2(n-1, 0)\, e^{j\bar{\omega}n}$, it will be possible, similarly as before, to confine ourselves in the approximate formula to a finite number of terms in the series. Namely, let us follow the reasoning set out below.

If we multiply both sides of Formula (102) by $z - 1$, we shall obtain

$$\begin{aligned} (z - 1) K_i^*(z, \varepsilon) &= (z - 1) x_2(0, \varepsilon) + \\ &+ (z - 1) \sum_{n=1}^{\infty} [x_2(n, \varepsilon) - x_2(n - 1, \varepsilon)] z^{-n} \end{aligned} \tag{112}$$

whence, after elementary transformations

$$\begin{aligned} (z - 1) K_i^*(z, \varepsilon) &= (z - 1) x_2(0, \varepsilon) + \Delta x_2(0, \varepsilon) + \\ &+ \sum_{n=1}^{\infty} \Delta [x_2(n, \varepsilon) - x_2(n - 1, \varepsilon)] z^{-n}, \end{aligned}$$

or

$$(z-1)K_i^*(z,\varepsilon) = (z-1)x_2(0,\varepsilon) + \Delta x_2(0,\varepsilon) + \\ + \sum_{n=1}^{\infty} \Delta^2 x_2(n-1,\varepsilon) z^{-n}, \tag{113}$$

where

$$\Delta^2 x_2(n-1,\varepsilon) = \Delta[x_2(n,\varepsilon) - x_2(n-1,\varepsilon)] = \\ = x_2(n+1,\varepsilon) - 2x_2(n,\varepsilon) + x_2(n-1,\varepsilon) \tag{114}$$

and

$$\Delta x_2(0,\varepsilon) = x_2(1,\varepsilon) - x_2(0,\varepsilon) \tag{115}$$

The frequency characteristic $K^*(e^{j\bar{\omega}}, 0)$ of an open sampled-data system will then be expressed by the formula

$$K^*(e^{j\bar{\omega}}, 0) = \frac{1}{e^{j\bar{\omega}} - 1} \sum_{n=1}^{\infty} \Delta^2 x_2(n-1, 0) e^{-j\bar{\omega}n} + \\ + x_2(0,0) + \frac{\Delta x_2(0,0)}{e^{j\bar{\omega}} - 1}. \tag{116}$$

Subsequently, using the relation (110), which is correct for an astatic characteristic, we may, in the above formula, drop all the terms with the indices $n > N$. We then obtain the approximate formula

$$K^*(e^{j\bar{\omega}}, 0) \approx \frac{1}{e^{j\bar{\omega}} - 1} \sum_{n=1}^{N} \Delta^2 x_2(n-1, 0) e^{-j\bar{\omega}n} + \\ + x_2(0,0) + \frac{\Delta x_2(0,0)}{e^{j\bar{\omega}} - 1}. \tag{117}$$

It is also possible to deduce formulae determining the components of the complex function $K^*(e^{j\bar{\omega}}, 0)$. Namely, taking into consideration the obvious relations

$$\mathcal{R}_e\left\{\frac{1}{e^{j\bar{\omega}} - 1}\right\} = -\frac{1}{2}; \quad \mathcal{I}_m\left\{\frac{1}{e^{j\bar{\omega}} - 1}\right\} = -\frac{1}{2}\cot\frac{\bar{\omega}}{2} \tag{118}$$

and using the formulae determining the components of the product of complex numbers, we shall obtain

$$\mathcal{R}_e K^*(e^{j\bar{\omega}}, 0) \approx x_2(0,0) - \frac{1}{2}\Delta x(0,0) + \\ - \frac{1}{2}\sum_{n=1}^{N} \Delta^2 x_2(n-1, 0)\cos\bar{\omega}n +$$

$$-\frac{1}{2}\cot\frac{\bar\omega}{2}\sum_{n=1}^{N}\Delta^2 x_2(n-1,0)\sin\bar\omega n, \tag{119}$$

$$\mathcal{I}_m K^*(e^{j\bar\omega},0)\approx\frac{1}{2}\sum_{n=1}^{N}\Delta^2 x_2(n-1,0)\sin\bar\omega n+$$

$$-\frac{1}{2}\cot\frac{\bar\omega}{2}\sum_{n=1}^{N}\Delta^2 x_2(n-1,0)\cos\bar\omega n+$$

$$-\frac{1}{2}\cot\frac{\bar\omega}{2}\Delta x_2(0,0). \tag{120}$$

In terms of the formulae obtained determining $\mathcal{R}_e K^*(e^{j\bar\omega},0)$ and $\mathcal{I}_m K^*(e^{j\bar\omega},0)$, from knowledge of the time characteristic $x_2(n,0)$ it is possible to find the plot of the Nyquist diagram of an open sampled-data control system containing an intergrating element (for example, a servomotor). Knowing the plot of this characteristic in the interval $\bar\omega\in[-\pi,\pi]$, we can determine the stability of the control system.

Concluding these considerations, it is worth noting that the knowledge of the Nyquist diagram $K^*(e^{j\bar\omega},0)$ may be useful not only in determining the stability of a control system. From the plot of the function $K^*(e^{j\bar\omega},0)$, we can draw conclusions concerning the degree of stability and we can anticipate what correction elements should be used for improving the quality of the control process.

3.2.3. Stability conditions of a static sampled-data control with a typical characteristic

The method presented in the preceding paragraph may be applied to any static or astatic sampled-data control system. In spite of its universality, this method has a certain drawback, in that it is necessary to perform preliminary calculations in order to investigate the stability of a system. It will be shown below that in the case in which the time characteristic $x_2(n,0)$ of an open sampled-data control system is typical, as indicated in Fig. 24a, the stability of the system may be determined directly from the plot of $x_2(n,0)$, without the necessity to determine the Nyquist diagram $K^*(e^{j\bar\omega},0)$. The reasoning is based on the substitution of a simplified characteristic in the form of a broken line in Fig. 24b for the characteristic from Fig. 24a.

The simplified characteristic may be represented analytically in the following manner

$$x_2(n, 0) = \begin{cases} 0; & n < n_1 \\ \dfrac{k}{n_2 - n_1}(n - n_1); & n_1 \leqslant n < n_2 \\ k; & n \leqslant n_2 \end{cases} \tag{121}$$

Since the input signal $x_2(t)$ is actuated by a unit-step excitation $\mathbf{1}(t)$, then by virtue of the formula presented above, the coefficient k is equal to the "resultant" amplification in the feedbeck loop.

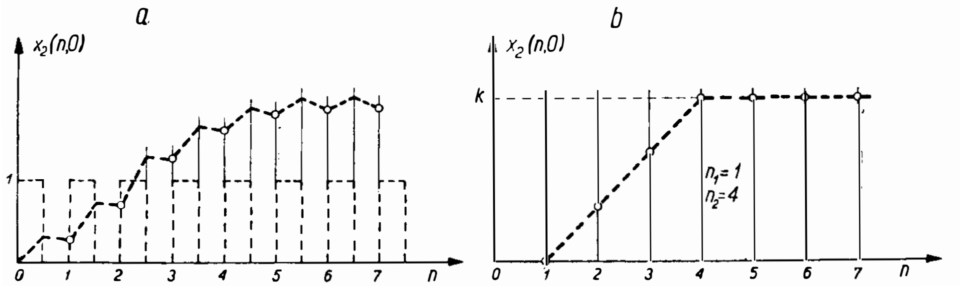


Fig. 24. Typical time characteristic of an open static sampled-data control system

Using the Laplace transformation, we may express the simplified characteristic $x_2(n, 0)$ by the formula

$$\mathcal{L}\{x_2(n, 0)\} = \frac{k}{n_2 - n_1} \cdot \frac{z}{(z-1)^2}[z^{-n_1} - z^{-n_2}]\,\eta \tag{122}$$

where $z = e^s$ is the displacement operator.

Since we have assumed that a unit-step excitation was applied to the input of the system — that is,

$$\mathcal{L}\{x_1(n)\} = \mathcal{L}\{1\} = \frac{z}{z-1}\eta \quad \bullet \tag{123}$$

then

$$K^*(z, 0) = \frac{\mathcal{L}\{x_2(n, 0)\}}{\mathcal{L}\{x_1(n, 0)\}} = \frac{k}{n_2 - n_1} \cdot \frac{z^{-n_1} - z^{-n_2}}{z - 1} \tag{124}$$

Substituting into the above formula, in place of the displacement operator z, the function e^{j-}, we shall obtain the expression $K^*(e^{j\bar{\omega}}, 0)$, which determines the frequency characteristic of the open control system

$$K^*(e^{j\omega}, 0) = \frac{k}{n_2 - n_1} \cdot \frac{e^{-j\bar{\omega} n_1} - e^{-j\omega n_2}}{e^{j\omega} - 1} \tag{125}$$

In accordance with the theorem cited in the preceding chapter, the stability limit of the system is determined from the equation

$$\begin{aligned} &\text{a)}\ \mathcal{R}_e K^*(e^{j\bar{\omega}}, 0) = -1, \\ &\text{b)}\ \mathcal{I}_m K^*(e^{j\bar{\omega}}, 0) = 0. \end{aligned} \tag{126}$$

For this purpose, we shall investigate the roots of the equation

$$K^*(e^{j\bar{\omega}}, 0) + 1 = 0 \tag{127}$$

Taking into consideration in this equation the expression (125), we shall arrive at

$$k(e^{-j\bar{\omega}n_1} - e^{-j\bar{\omega}n_2}) + (n_2 - n_1)(e^{j\bar{\omega}} - 1) = 0 \tag{128}$$

and hence, after performing considerable elementary trigonometric transformations, we shall obtain the following relations (c) and (d)

$$\begin{aligned} &\text{c)}\ 2k \sin\bar{\omega}\alpha \sin\bar{\omega}\beta - (n_2 - n_1)(\cos\bar{\omega} - 1) = 0, \\ &\text{d)}\ 2k \cos\bar{\omega}\alpha \sin\bar{\omega}\beta - (n_2 - n_1)\sin\bar{\omega} = 0, \end{aligned} \tag{129}$$

where

$$\alpha = \frac{n_1 + n_2}{2}, \qquad \beta = \frac{n_1 - n_2}{2}$$

The above relations will be considered as a set of two equations with two unknowns $\bar{\omega}$ and k. Dividing Equations (c) and (d) one by the other, we shall find the unknown $\bar{\omega}$

$$\tan\bar{\omega}\alpha = \frac{\cos\bar{\omega} - 1}{\sin\bar{\omega}} = -\tan\frac{\bar{\omega}}{2} \tag{130}$$

hence,

$$\bar{\omega}\alpha = -\frac{\bar{\omega}}{2} + (\nu + 1)\pi; \qquad \nu = 0; \qquad \pm 1; \qquad \pm 2; \ldots$$

and

$$\bar{\omega} = \bar{\omega}_\nu = \frac{(\nu + 1)2\pi}{1 + n_1 + n_2} \tag{131}$$

Then, for example from Equation (d), we shall determine the unknown k

$$k = k_\nu = (n_2 - n_1)\frac{\sin\bar{\omega}_\nu}{2\cos\bar{\omega}_\nu\alpha \sin\bar{\omega}_\nu\beta} =$$

$$= (n_2 - n_1)\frac{\sin\dfrac{(\nu + 1)2\pi}{1 + n_1 + n_2}}{2\cos\dfrac{n_1 + n_2}{1 + n_1 + n_2}(\nu + 1)\pi \sin\dfrac{n_1 - n_2}{1 + n_1 + n_2}(\nu + 1)\pi} \tag{132}$$

This formula can be simplified. Namely, if we take into consideration the obvious relation

$$\frac{n_1 + n_2}{1 + n_1 + n_2} = 1 - \frac{1}{1 + n_1 + n_2} \tag{133}$$

whence

$$\cos \frac{n_1 + n_2}{1 + n_1 + n_2} (\nu + 1)\pi = (-1)^{\nu+1} \cos \frac{(\nu + 1)\pi}{1 + n_1 + n_2} \tag{134}$$

and then if we introduce the transformation

$$\sin \frac{(\nu + 1)2\pi}{1 + n_1 + n_2} = 2 \sin \frac{(\nu + 1)\pi}{1 + n_1 + n_2} \cos \frac{(\nu + 1)\pi}{1 + n_1 + n_2} \tag{135}$$

we shall obtain a simple formula defining k_ν

$$k_\nu = (-1)^\nu (n_2 - n_1) \frac{\sin \frac{(\nu + 1)\pi}{1 + n_1 + n_2}}{\sin \frac{n_2 - n_1}{1 + n_1 + n_2} (\nu + 1)\pi} \tag{136}$$

This formula is ambiguous and determines the stability conditions for the particular components of the control process. Since we are considering a system with a negative feedback, of interest to us in the formula obtained are only those values of the number ν which yield positive values of the coefficient k_ν, with n_1 and n_2 as assumed. Thus, for example, we should not take into consideration those values of ν for which the expression

$$\frac{\nu + 1}{1 + n_1 + n_2} \tag{137}$$

is an integral number. For we can verify that then $k_\nu = -1$.

For the fundamental component k_0, we shall obtain the formula

$$k_0 = (n_2 - n_1) \frac{\sin \frac{\pi}{1 + n_1 + n_2}}{\sin \frac{n_2 - n_1}{1 + n_1 + n_2} \pi} \tag{138}$$

It can be shown that this formula gives the lowest value of the coefficient $k_\nu > 0$. In view of the above, the Formula (138) determines the greatest admissible amplification in the feedback loop. For the amplification $k > k_0$, the system is unstable. Figure 25 shows the plots of the function

$$k_0 = f(n_1, n_2) \tag{139}$$

where n_1 is treated as a parameter and n_2 — as an independent variable. Making use of these plots, we can easily determine the stability of a static sampled-data control system from knowledge of the time characteristic

$x_2(n, 0)$ of the open system. Note that k_0 is not a function of the ratio of the "delay" time to the "steady-state" time $\frac{n_1}{n_2}$ in the waveform of the time characteristic, but constitutes a function of two variables n_1 and n_2. From the plot of the function $k_0 = f(n_1, n_2)$, it follows, with n_1 assumed, that the greater n_2 — that is, the slower the time characteristic of the open system increases — the greater the amplification it is possible

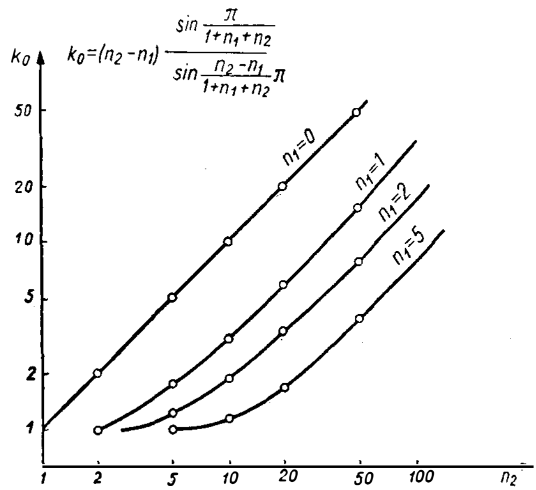


Fig. 25. Plots of the function $k_0 = f(n_1, n_2)$

to apply. Of course, we are able to influence the speed of the increase of the time characteristic x_2(n, 0) by changing the width γ of the rectangular pulses generated by the sampler.

The method presented for determining stability does not require any preliminary calculations.

3.2.4. Stability conditions for an astatic sampled-data control system with a typical characteristic

Below, we shall discuss the stability of an astatic sampled-data control system. The analysis performed is based on the assumption that the waveform of the time characteristic $x_2(n, 0)$ (response to a unit-step excitation) has the shape indicated in Fig. 26a. Such a characteristic can be obtained experimentally, or can be determined in a simple manner by the analytical-graphical method.

Note, that, as regards astatic systems, the stability of a system depends only on the plot of the characteristic $x_2(n, \varepsilon)$ at the moments

$n = 0, 1, 2, \ldots$ and $\varepsilon = 0$, — that is, the plot of the characteristic $x_2(n, 0)$. The behavior of the characteristic $x_2(n, \varepsilon)$ between the points $n = 0, 1, 2, \ldots$ has absolutely no bearing on the stability of the system. Thus the method presented may be used for a relatively large class of astatic sampled-data control systems encountered in practice.

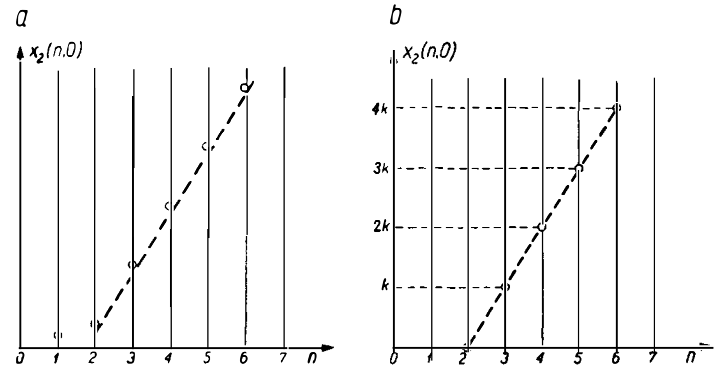


Fig. 26. Typical time charateristic of an open astatic sampled-data control system (a), and the simplified characteristic (b)

In our considerations, we shall replace the characteristic $x_2(n, 0)$ (Fig. 26a) by a simplified characteristic in the form of a straight line (Fig. 26b); we shall then assume that

$$x_2(n, 0) = \begin{cases} 0; & n < n_0 \\ k(n - n_0); & n \geqslant n_0. \end{cases} \tag{140}$$

Since we have

$$\mathfrak{L}\{n\} = \frac{z}{(z-1)^2}\eta \tag{141}$$

there will be

$$\mathfrak{L}\{x_2(n, 0)\} = k\frac{z}{(z-1)^2}z^{-n_0}\eta$$

and

$$K^*(z, 0) = \frac{z-1}{z\eta}\mathfrak{L}\{x_2(n, 0)\} = k\frac{z^{-n_0}}{z-1} \tag{142}$$

The frequency characteristic of an open control system (corresponding to the moments $\varepsilon = 0$) will be obtained by substituting for the displacement operator z the function $e^{j\bar{\omega}}$

$$K^*(e^{j\omega}, 0) = k\frac{e^{-j\omega n_0}}{e^{j\omega} - 1} \tag{143}$$

Similarly as in the preceding chapter, the stability limit will be determined from the conditions

$$\begin{aligned} &\text{(a)}\ \mathcal{R}_e K^*(e^{j\bar{\omega}}, 0) = -1\,, \\ &\text{(b)}\ \mathcal{I}_m K^*(e^{j\bar{\omega}}, 0) = 0 \end{aligned} \tag{144}$$

Taking into consideration (143), we shall obtain

$$\mathcal{R}_e K^*(e^{j\bar{\omega}}, 0) = \mathcal{R}_e \left\{k \frac{e^{-j\bar{\omega}n_0}}{e^{j\bar{\omega}} - 1}\right\} =$$

$$= \frac{k}{(\cos\bar{\omega} - 1)^2 + \sin^2\bar{\omega}} [\cos\bar{\omega}n_0 (\cos\bar{\omega} - 1) - \sin\bar{\omega}n_0 \sin\bar{\omega}]\,,$$

$$\mathcal{I}_m K^*(e^{j\bar{\omega}}, 0) = \mathcal{I}_m \left\{k \frac{e^{-j\omega n_0}}{e^{j\bar{\omega}} - 1}\right\} =$$

$$= -\frac{k}{(\cos\bar{\omega} - 1)^2 + \sin^2\bar{\omega}} [\cos\bar{\omega}n_0 \sin\bar{\omega} + \sin\bar{\omega}n_0 (\cos\bar{\omega} - 1)]$$

The conditions (a) and (b) lead then to the following equalities (c) and (d)

$$\begin{aligned} &\text{(c)}\ k\,[\sin\bar{\omega}\sin\bar{\omega}n_0 - (\cos\bar{\omega} - 1)\cos\bar{\omega}\,n_0] = (\cos\bar{\omega} - 1)^2 + \sin^2\bar{\omega}\,; \\ &\text{(d)}\ \sin\bar{\omega}\cdot\cos\bar{\omega}n_0 + (\cos\bar{\omega} - 1)\sin\bar{\omega}n_0 = 0\,. \end{aligned} \tag{145}$$

After considerable elementary trigonometric transformations, we shall arrive at a simpler equation — namely

$$\begin{aligned} &\text{(c}'\text{)}\ k\sin\left(\bar{\omega}n_0 + \frac{\bar{\omega}}{2}\right) = 2\sin\frac{\bar{\omega}}{2}\,, \\ &\text{(d}'\text{)}\ \cos\left(\bar{\omega}n_0 + \frac{\bar{\omega}}{2}\right) = 0 \end{aligned} \tag{146}$$

These equations will be solved with respect to the unknown $\bar{\omega}$ and k. From (d′) it follows that

$$\bar{\omega}n_0 + \frac{\bar{\omega}}{2} = \frac{\pi}{2} + \nu\pi$$

where ν is integral number. Hence

$$\bar{\omega} = \bar{\omega}_\nu = \frac{1 + 2\nu}{1 + 2n_0}\pi$$

Substitung $\bar{\omega}_\nu$ into Equation (c′), we shall then find the unknown k

$$k = k_\nu = \frac{2\sin\dfrac{\bar{\omega}}{2}}{\sin\left(\bar{\omega}n_0 + \dfrac{\bar{\omega}}{2}\right)} = (-1)^\nu\, 2\sin\frac{1 + 2\nu}{1 + 2n_0}\cdot\frac{\pi}{2} \tag{147}$$

It can easily be shown that in this case also, the smallest non-negative value of k_ν will be obtained for the fundamental component of the control process — that is, for $\nu = 0$. The greatest adsmissible inclination of the time characteristic $x_2(n, 0)$ of an open control system is then k_0

$$k_0 = 2 \sin \frac{1}{1 + 2n_0} \cdot \frac{\pi}{2} \tag{148}$$

The function $k_0 = f(n_0)$ is shown in Fig. 27. From the plot of this function, we conclude that the greatest admissible inclination of the time characteristic of an open sampled-data control system decreases with increase in n_0 — that is, with increase in the delay introduced by the system.

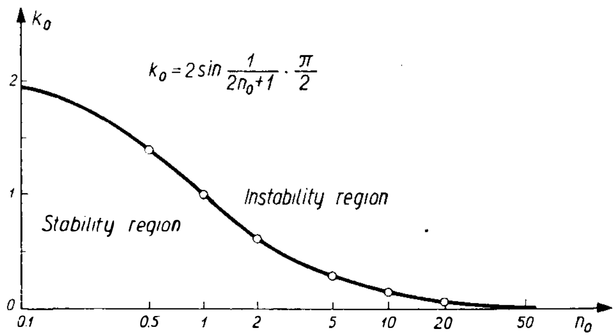


Fig. 27. Plot of the function $k_0 = f(n_0)$, determining the greatest admissible amplification in the feedback loop of an astatic sampled-data control system

The method presented for determining stability is convenient primarily when the time characteristic $x_2(n, 0)$ of an open system is determined experimentally. If, for some reason, experimental determination of the above characteristic is impossible, the characteristic can be calculated analytically — for example by means of the method presented in Part 2. Then we use the equation of a sampled-data system

$$\mathcal{D}\{x_2(n, 0)\} = K_I(z, 0)\, \mathcal{D}\{x_1(n)\}$$

and we assume that

$$\mathcal{D}\{x_1(n)\} = \mathcal{D}\{1\} = \frac{z}{z-1}$$

The characteristic $x_2(n, 0)$ of an open sampled-data control system may also be determined by the graphical-analytical method; namely, we may add graphically the responses $h_\gamma(t) = h(t) \mp h(t - \gamma T)$ of the linear part of the system to rectangular pulses of width γT, which are displaced with respect one to another by the period T.

It is clear that an influence on the inclination of the characteristic $x_2(n, 0)$ is exerted by changing the width of the rectangular pulses — that is, by a change in the parameter γ. It can easily be seen that the inclination of $x_2(n, 0)$ decreases with a decrease in the parameter γ.

Simple astatic systems and their time characteristics $x_2(\bar{t})$, $x_2(n, 0)$ are presented in Fig. 28 and 29.

Note that in the case of the system from Fig. 28, between the inclination k_0 of the characteristic $x_2(n, 0)$ and the width of the pulses, a simple relation holds

$$k_0 = \frac{\gamma}{T_1}.$$

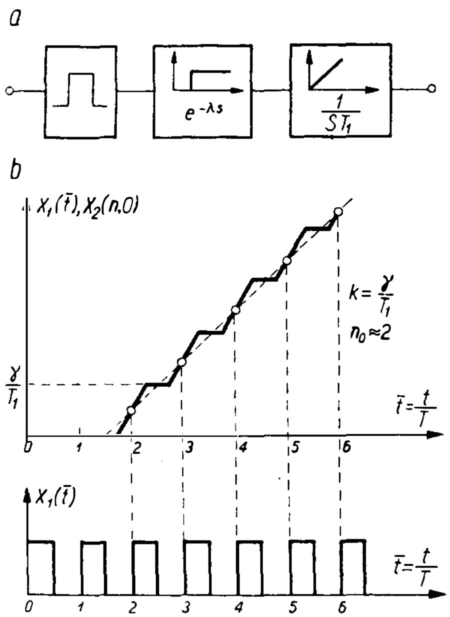


Fig. 28. Time characteristic of an astatic system containing a delay element

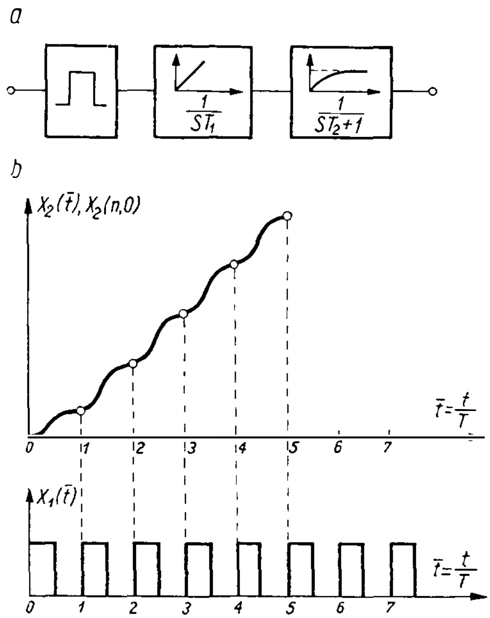


Fig. 29. Time characteristic of an astatic system containing an inertial element

Using this formula, we can find the greatest admissible width of pulses for a system with the characteristic shown in Fig. 28. The control system of gas pressure [4] may serve as an example for a system with such a characteristic. An example for a system with the characteristic shown in Fig. 29 may be the temperature control system [1].

4.1. Conditions of finite transients

An interesting property of sampled-data feedback systems is the possibility of obtaining a finite time for the passing of a time characteristic into a steady state. As we know, linear continuous-data systems do not possess this property. By way of ilustration, let us consider the pulse-transfer function of a sampled-data feedback system

$$K^*_{s,i}(e^s, \varepsilon) = \frac{K^*_i(e^s, \varepsilon)}{1 + K^*_l(e^s, 0)} = \frac{K^*_i(e^s, \varepsilon)}{\varphi^*(e^s)},$$

where K^*_i is the transfer function of an open-loop feedback system.

If we assume that the system contains only lumped elements, then K^*_i will be a rational function, that is

$$K^*_l(e^s, 0) = \frac{M^*(e^s)}{D^*(e^s)}$$

where M^* and D^* are polynomials of e^s.

The function φ^* will then also be a rational function, since we have

$$\varphi^*(e^s) = 1 + \frac{M^*(e^s)}{D^*(e^s)} = \frac{D^*(e^s) + M^*(e^s)}{D^*(e^s)} = \frac{G^*(e^s)}{D^*(e^s)}$$

The stability of a feedback system evidently depends on the roots of the equation

$$G^*(e^s) = D^*(e^s) + M^*(e^s) = 0$$

If the system under consideration is stable, then all the roots s_ν of the above equation have negative real parts. Let us assume that

$$\xi = \min|\mathcal{R}e\, s_\nu| \tag{149}$$

The number ξ is called the degree of stability of a system. It turns out that in the case of sampled-data systems, we can attain an infinite degree of stability — that is, with suitable assumptions we can realize a sampled-data system for which $\xi = \infty$.

In fact, if we assume that the function $G^*(e^s)$ is a polynomial of the order l, that is

$$G^*(e^s) = a_l e^{sl} + a_{l-1} e^{s(l-1)} + \ldots + a_0$$

then for the case

$$a_0 = a_1 = \ldots = a_{l-1} = 0; \qquad \alpha_l \neq 0 \tag{150}$$

we shall obtain

$$G^*(e^s) = D^*(e^s) + M^*(e^s) = a_l e^{sl}$$

and

$$\xi = |\mathcal{R}_e\, s_\nu| = \infty$$

Since the transfer function K^* of an open control system is the quotient of the polynomials M^* and D^* — that is

$$K^* = \frac{M^*}{D^*} = \frac{b_0' + b_1' e^s + \ldots + b_{l_1}' e^{sl_1}}{a_0' + a_1' e^s + \ldots + a_l' e^{sl}}; \qquad l_1 \leqslant l$$

then the condition (150) imposes the following conditions on the coefficients of the transfer function K^* of an open control system

$$\left.\begin{array}{c} a_0' = -b_0';\ a_1' = -b_1' \ \ldots a_{l-1}' = -b_{l-1}' \\ \text{and} \qquad a_l' \neq b_l'. \end{array}\right\} \tag{151}$$

Satisfying these conditions, we shall obtain

$$G^* = M^* + D^* = (a_l' + b_l')\, e^{sl},$$

thus, in fact, for this case

$$\xi = \min|\mathcal{R}_e\, s_\nu| = \infty$$

We shall prove now the following.

Property. In the case of an infinite degree of stability, the time characteristic of a closed sampled-data control system has a finite transient.

Proof. In fact, since the pulse transfer function of a closed control system is a function taking the form

$$K_{s,i}^*(e^s, \varepsilon) = \frac{b_0(\varepsilon) + b_1(\varepsilon) e^s + \ldots + b_k(\varepsilon) e^{sk}}{a_0 + a_1 e^s + \ldots + a_l e^{sl}}; \qquad l \geqslant k$$

then with the assumption that

$$a_0 = a_1 = \ldots = a_{l-1} = 0; \qquad a_l \neq 0$$

we shall obtain

$$K_{s,i}^*(e^s, \varepsilon) = \frac{1}{a_l} e^{-sl} [b_0(\varepsilon) + b_1(\varepsilon) e^s + \ldots + b_k(\varepsilon) e^{sk}] =$$

$$= \frac{1}{a_l} [b_0(\varepsilon) e^{-sl} + b_1(\varepsilon) e^{-(l-1)s} + \ldots + b_k(\varepsilon) e^{-s(l-k)}]; \qquad l - k \geqslant 0$$

and

$$x_2[\bar{t}, \varepsilon] = \mathcal{L}^{-1}\left\{K_{s,i}^*(e^s, \varepsilon)\frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{b_0(\varepsilon)}{a_l} \cdot \frac{e^{-sl}}{s} + \ldots + \frac{b_k(\varepsilon)}{a_1} \cdot \frac{e^{-(l-k)s}}{s}\right\}$$

Since generally we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\lambda s}}{s}\right\} = \begin{cases} 0; & \bar{t} < \lambda \\ 1; & \bar{t} > \lambda \end{cases}$$

then the function $x_2[\bar{t}, \varepsilon]$ for $\bar{t} \geqslant l$ is a steady-state function. Quod erat demonstrandum.

4.2. Analogies with integral criteria

A suitable choice of parameters for a sampled-data system such that the control process will have a finite period of duration is not always feasible. In order to satisfy this condition, it is most often necessary to apply special correction systems or differentiating elements, which are difficult as regards realization; moreover, they raise the cost of equipment and are not always desirable in a system. In many cases, the application of special correction systems, which makes the structure of the system more complicated, would not be desirable or justified. Thus the question often arises as to what, so that the "best" control process be obtained, should be the parameters of a system with a structure given in advance.

It is clear that the use of the term "the best" would make no sense without a precise definition of this term. Accordingly, we make use of suitably defined quality criteria of the control process. These criteria are analogous to the integral criteria used in the problems of continuous--data control. Namely, we use the expressions

$$I_1 = \sum_{n=0}^{\infty} |x_2(n) - x_2(\infty)| \tag{152a}$$

$$I_{1,n} = \sum_{n=0}^{\infty} n\,|x_2(n) - x_2(\infty)| \tag{152b}$$

$$I_2 = \sum_{n=0}^{\infty} [x_2(n) - x_2(\infty)]^2 \tag{152c}$$

where $x_2(n)$ is a sequence formed from the functions of the output signal of the system.

It is worth noting that the expression (152a) — that is the quantity I_1, is the measure of the field contained between the step function determined by the sequence $x_2(n)$, and the steady-state value $x_2(\infty)$ of the function x_2.

The use of the expressions I_1, and $I_{1,\,n}$ is convenient in calculus, only when the terms of the sequence

$$x^2(n) - x_2(\infty)$$

have the same sign; for this case the relation

$$|x_2(n) - x_2(\infty)| = \pm\,(x_2(n) - x_2(\infty))$$

holds, and in the definition formulae (152a) and (152b), we may drop the sign of the modulus. However, when $x_2(n)$ approaches in an oscillative manner the asymptotic value $x_2(\infty)$ the sign of the modulus cannot be dropped, and therefore the calculation of the expression I_1 and $I_{1,n}$ becomes much more complicated. Thus it is better in this case, to use the expression I_2.

The quantities I_1, $I_{1,n}$ and I_2 are calculated directly from the pulse-transfer function $K^*(e^s, 0)$ of the closed system. Namely, taking into consideration the relation

$$\mathcal{L}\{x_2[\bar{t}] - x_2[\infty]\} = [K^*(e^s, 0) - K^*(1, 0)]\frac{e^s}{e^s - 1}\eta$$

and Formula (25), we obtain for the quantity I_1 the following expression

$$I_1 = \lim_{s \to 0}[K^*(e^s, 0) - K^*(1, 0]\frac{e^s}{e^s - 1} \qquad (153)$$

In a similar way (using Formula 24), we shall obtain

$$I_{1,n} = -\lim_{s \to 0}\frac{d}{de^s}[K^*(e^s, 0) - K^*(1, 0)]\frac{e^s}{e^s - 1}. \qquad (154)$$

The quantity I_2 can also be calculated directly from knowledge of the transfer function of a closed-loop sampled-data feedback system. The deduction of a suitable formula relating the quantity I_2 to the transfer function of the system involves in this case somewhat more serious difficulties. In a particular case, in which

$$[K^*(e^s, 0) - K^*(1, 0)]\frac{e^s}{e^s - 1} = \frac{d_2 e^{2s} + d_1 e^s}{a_2 e^{2s} + a_1 e^s + a_0}$$

the quantity I_2 are determined from the formula

$$I_2 = \frac{(d_2^2 + d_1^2)(a_2 + a_0) - 2d_1 d_2 a_1}{(a_2 - a_0)[(a_2 + a_0)^2 - a_1^2]} \qquad (155)$$

However, in the case in which

$$[K^*(e^s, 0) - K^*(1, 0)]\frac{e^s}{e - 1} = \frac{d_1 e^s}{a_1 e^s + a_0}$$

we use a simpler formula — namely,

$$I_2 = \frac{d_1^2}{a_1^2 - a_0^2} \qquad (155a)$$

For the deducting of the formulae given above, the reader is referred to the paper [1].

5. EXAMPLE OF DESIGNING A SAMPLED-DATA CONTROL SYSTEM

In order to design a sampled-data control system, a number of operations must be performed — namely:

(1) "to translate" the actual system into an equivalent block diagram, to assume a suitable structure of the controller, and to determine the transfer function $K(s)$ of the particular elements of the control system;

(2) to determine the pulse transfer functions $K_i^*(e^s, \varepsilon)$ of the open control system;

(3) to investigate the stability of the system;

(4) to determine the optimal parameters of the controller;

(5) to find the time characteristic of the open control system.

The manner of designing a sampled-data control system is illustrated by the following example.

Example. Figure 30 represents a diagram of a typical sampled-data control system for temperature. This system possesses, in addition to a controlled system, a sampler, a performing element, and a measurement

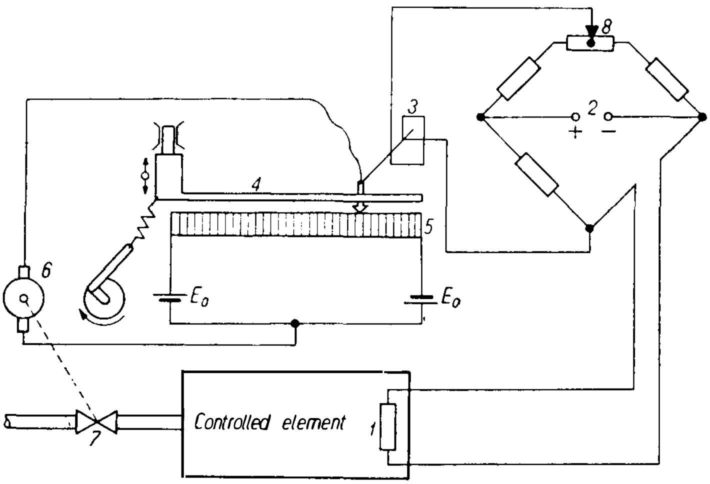


Fig. 30. Typical sampled-data control system for temperature (1) measuring element, (2) measurement bridge, (3) galvanometer frame, (4) sampler handle, (5) resistor of the sampler, (6) servomotor, (7) controlling valve, (8) unit regulating temperature

element. A measurement element is the servomotor which changes the flow of heat by means of changes in the position of the valve *7*. A measurement element is the resistor *1*, the resistance of which depends on temperature and which constitutes one of the arms of the measurement bridge *2*. The desired temperature is fixed by means of the potentiometer *8*, situated in the bridge loop.

The output voltage of the bridge, which is of course proportional to the error — that is, to the difference between the given temperature and the temperature subjected to control — is measured by means of the galvanometer *3*.

Owing to the presence of a sampler, the servomotor and thus also the valve controlling the flow of heat, works in a non-continuous manner. The sampler, as well as the pulse action, acts at the same time as an amplifier.

Let us pass now to the performance of the first of the operations listed above which are connected with the design of a sampled-data control system.

Determination of the block diagram and the transfer functions for the particular elements of the control system

The dynamic of a system may be determined by means of the operator transfer functions of the particular elements. The servomotor may — as we know — be considered approximately as an integrating element; that is, the transfer function of the servomotor can be defined by the formula

$$K_i(s) = \frac{1}{T_s s}$$

where T_s is a constant of the servomotor.

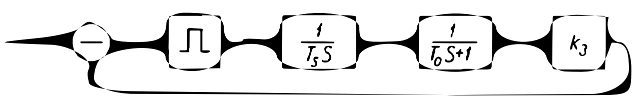


Fig. 31. Block diagram of a control system for temperature

The transfer function of the controlled system, which is for example a furnace or a thermostat, may approximately be considered as a first-order element; that is to say, for the controlled system we may assume that

$$K_2(s) = \frac{1}{T_0 s + 1}$$

where T_0 is a time constant of the system.

For the measurement element — which approximately does not contain inertial elements — we may assume that

$$K_3(s) = k_3$$

The block diagram, in accordance with the above considerations, takes the form as given in Fig. 31.

Determination of the pulse-transfer function of an open control system

The transfer function $K(s)$ of the linear part of an open control system — that is, of a system constituting a series connection of the elements K_1, K_2 and K_3, is the product

$$K(s) = K_1(s) K_2(s) K_3(s) = \frac{k_3}{T_s s (T_0 s + 1)} \tag{156}$$

The linear part of an open control system is then an integral-inertial element. Using Table 1 (See Appendix), we immediately find the pulse-transfer functions corresponding to the expression (156). Namely, we have

$$K_I^*(e^s, \varepsilon) = -k_0 \frac{\beta_s}{\beta} + k_0 \beta_s \left(\varepsilon + \frac{\gamma}{e^s - 1} + k_0 \frac{\beta_s}{\beta} \cdot \frac{e^s - e^{-\beta(1-\gamma)}}{e^s - e^{-\beta}} e^{-\beta\varepsilon}\right) \tag{157a}$$

$$\text{for } 0 \leqslant \varepsilon < \gamma$$

$$K_{II}^*(e^s, \varepsilon) = k_0 \beta_s \gamma \frac{e^s}{e^s - 1} - k_0 \frac{\beta_s}{\beta} (e^{\beta\gamma} - 1) \frac{e^s}{e^s - e^{-\beta}} e^{-\beta\varepsilon} \tag{157b}$$

$$\text{for } \gamma \leqslant \varepsilon < 1$$

where

$k_0 = k \cdot k_3$, and k is the amplification of the sampler, $\beta = \frac{T}{T_0}$, $\beta_s = \frac{T}{T_s}$

and T is the period of rectangular pulses generated by the sampler.

Investigation of the stability of the system

We investigate the stability of the system in terms of frequency criteria. Since the stability of a system is entirely dependent on the behavior of the system at the moments $\varepsilon = 0$, then in order to investigate stability, we determine the quantity $K_I^*(e^s, 0)$ from Formula (157a). We have

$$K_I^*(e^s, 0) = -k_0 \frac{\beta_s}{\beta} + k_0 \beta_s \frac{\gamma}{e^s - 1} + k_0 \frac{\beta_s}{\beta} \cdot \frac{e^s - e^{\beta(1-\gamma)}}{e^s - e^{-\beta}} \tag{158}$$

The pulse frequency characteristic corresponding to the case $\varepsilon = 0$ then takes the form

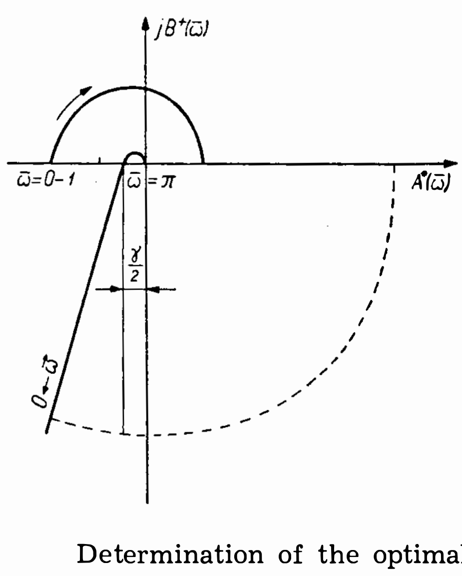
$$K_I^*(e^{j\bar{\omega}}, 0) = k_0 \beta_s \left[- \frac{e^{\beta\gamma} - 1}{\beta} \cdot \frac{e^{-\beta}}{e^{j\bar{\omega}} - e^{-\beta}} + \frac{\gamma}{e^{j\bar{\omega}} - 1} \right] \tag{159}$$

It may be noted that the first in parenthesis of Formula (159) — that is, the expression

$$- \frac{e^{\beta\gamma} - 1}{\beta} \cdot \frac{e^{-\beta}}{e^{j\bar{\omega}} - e^{-\beta}}$$

— constitutes for $0 \leqslant \bar{\omega} \leqslant \pi$ the equation of the semicircle situated in the upper semiplane (Fig. 32), while the second term — that is $\frac{\gamma}{e^{j\bar{\omega}} - 1}$ — represents for $0 \leqslant \bar{\omega} \leqslant \pi$ the semiaxis passing through the point $\left(-\frac{\gamma}{2} + j0\right)$.

A necessary and sufficient condition for the system under consideration to be stable after the feedback loop has been closed, is the requirement that the Nyquist diagram $K^*(e^{j\bar{\omega}}, 0)$ shall not contain the point $(-1 + j0)$.



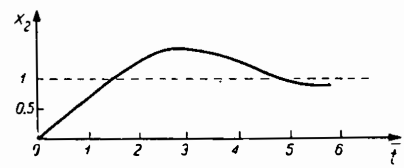


Fig. 32. Nyquist diagram $K^*(e^{j\bar{\omega}}, 0) = M^*(j\bar{\omega}, 0) = A^*(j\bar{\omega}) + jB^*(\bar{\omega})$ of an open sampled-data control system

Fig. 33. Time characteristic of a control system for temperature in the case in which $\beta = 0.5$ and $k_0\beta_s\gamma = 1.61$

Determination of the optimal parameters of the controller

In order to determine the optimal parameters of the controller, we use the formula

$$I_2 = \sum_{n=0}^{\infty} [x_2(n) - x_2(\infty)]^2$$

where x_2 is the unit-step response of the closed control system. We are taking into consideration the case in which $\gamma \ll 1$.

Since

$$\mathcal{L}\{x_2[\bar{t}, \varepsilon]\} = K_s^*(e^s, 0)\frac{e^s}{e^s - 1}\eta = \frac{K^*}{1 + K^*} \cdot \frac{e^s}{e^s - 1}\eta$$

and

$$\mathcal{L}\{x_2[\bar{t}, 0] - x_2[\sim]\} = [K_s^*(e^s, 0) - K_s^*(1, 0)]\frac{e^s}{e^s - 1}\eta =$$

$$= -\frac{e^{2s} - e^{-\beta} e^s}{e^{2s} - [(1 + e^{-\beta}) - k_0 \beta\gamma(1 - e^{-\beta})]e^s + e^{-\beta}}\eta$$

then in accordance with (155)

$$I_2 = \frac{(1 + e^{-2\beta})(1 + e^{-\beta}) - e^{-\beta}[1 + e^{-\beta} - k_0 \beta_s \gamma (1 - e^{-\beta})]}{(1 - e^{-\beta})\{(1 + e^{-\beta})^2 - [k_0 \beta_s \gamma (1 - e^{-\beta}) - (1 + e^{-\beta})]^2\}}$$

After introducing the denotations

$$a = 2\frac{1 + e^{-\beta}}{1 - e^{-\beta}} = 2\coth\frac{\beta}{2}; \quad b = k_0 \beta_s \gamma$$

the expression determining I_2 will be written as

$$I_2 = \frac{4(a - b) + a^2 b}{8(a - b)b} = \frac{1}{2b} + \frac{a^2}{8(a - b)}$$

Calculating the zero value of the derivative I_2 with respect to the parameter b

$$\frac{\partial I_2}{\partial b} = 0$$

we can then determine the optimal quantity of the product $k_0\beta_s\gamma$ for which the deviation of the control waveform from the unit-step function reaches its least value in the sense of the criterion of the squared error. After elementary calculations, we arrive at

$$(k_0 \beta_s \gamma)_{opt} = = 1 + e^{-\beta}$$

Determination of the time characteristic of the closed control system

The time characteristic of the closed control system is determined on the basis of the equation

$$\mathcal{L}\{x_2[\bar{t}, \varepsilon]\} = K_{s,i}^*(e^s, \varepsilon)\,\mathcal{L}\{x_0[\bar{t}]\}$$

Namely, if we assume that

$$x_0[\bar{t}] = \mathbf{1}(t)$$

then

$$\mathfrak{L}\{x_0[\bar{t}]\} = \frac{e^s}{e^s - 1}\eta$$

and

$$\mathfrak{L}\{x_2[t, \varepsilon]\} = K^*_{s,i}(e^s, \varepsilon)\frac{e^s}{e^s - 1}\eta = \frac{P^*(e^s, \varepsilon)}{Q^*(e^s)} \cdot \frac{e^s}{e^s - 1}\eta$$

where the functions P^* and Q^* are polynomials of the variable e^s.

By expanding the rational function into simple fractions, and reading from tables the results of the particular operations, we shall obtain the function $x_2[\bar{t}, \varepsilon]$ determining the time characteristic of the closed control system. The plot of the function $x_2[\bar{t}, \varepsilon]$ for the case $\beta = 0{,}5$ and $k_0\beta_s\gamma = 1{,}61$ is given in Fig. 33.

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ON SYNTHESIS OF SAMPLED — DATA CONTROL SYSTEMS [1]

N93-36011

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A sythesis method of sampled-data control systems on the basis of the frequency pattern in presented in the present paper. The method makes use of the properties of frequency and time characteristics concerning a sampled-data control system as discussed and established herein.

Author

1. INTRODUCTION

A control system is a negative feedback system. In such a system two essential elements can be distinguished — a controlled system and a controller.

An essential characteristic of a sampled-data control system is the presence of *a sampler in the controller*; the sampler transforms the input signal — usually a continuous function — into a signal in the form of sequence of rectangular pulses (Fig. 1).

Depending on the design of the sampler, we distinguish several types of sampled-data control systems. In our considerations, we shall confine ourselves to the investigation of systems with samplers transforming the input signal into a signal in the form of a sequence of pulses with modulated amplitudes or with a modulated width of pulses (Fig. 2).

The properties of the first type of sampler can be uniquely determined by means of three independent parameters:

T — period of occurrence of pulses,

k — amplification,

$\gamma = \frac{d}{T}$ — relative width of pulses,

where d is the width of pulses measured in seconds.

The properties of the second type of sampler are determined by

k — amplitude of generated pulses,

T — period of occurrence of pulses,

$\varkappa$ — width coefficient of pulses.

The sampled-data control systems of the first and the second type are classified in the category of dynamical systems, since certain parameters (amplitude or width of pulses) of the signal correcting the con-

[1] Archiwum Automatyki i Telemechaniki (Vol. II, No. 1-2, 1957, pp. 95—120)

trolled quantity are linearly dependent on the value of the actuating signal.

Sample-data control systems are of particular importance to the control of slowly changing processes, such as the control of pressure and temperature in boilers, temperature in industrial furnaces, etc.

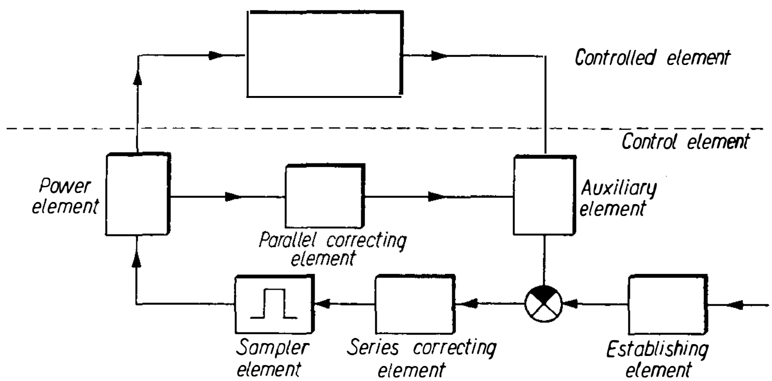

Fig. 1. Block diagram of a sampled-data control system

Moreover, such systems are widely applied in telemetering and remote control.

The methods of investigating control systems are usually divided into two groups — *analysis methods and synthesis methods.*

Synthesis methods aim at **determining the optimum structure and characteristics of the individual elements of the control system in terms of its properties given in advance.**

Analysis methods set out **to determine the properties of a system which are of interest to us, the structure and characteristics of elements being assumed in advance.**

In the development of analysis methods of sampled-data control systems, three stages may, in principle, be distinguished. First, such systems were investigated by the graphical-analytical method, then by a method which might be called a classical method based on the theory of difference equations, and finally by the method based on the so-called *discrete* (summing) Laplace transformation.

The last of the above mentioned methods, which was presented by Tsypkin [1] is particularly useful in engineering applications, since it gives theoretical bases common to a wide class of sampled-data control systems, and moreover, it shows promising possibilities of further development.This method employs a number of notions well known from the theory of dynamical systems — such as, for example, the notions of frequency and time characteristics, transfer function etc. — which fact makes it still more valuable.

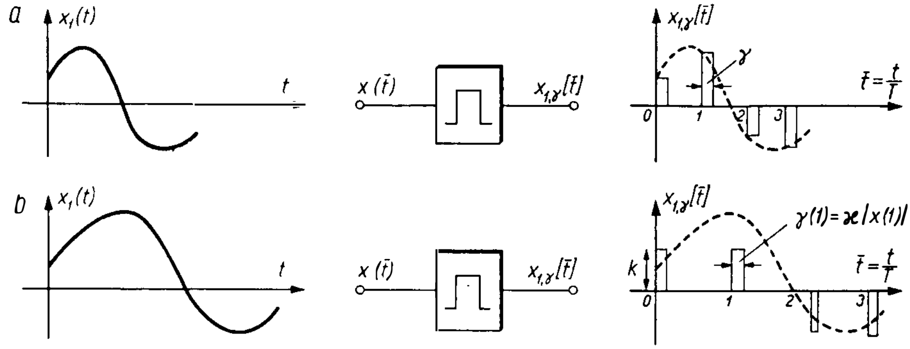

Fig. 2. Operation principle of a sampler (a) type 1, (b) type 2

In the opinion of the present author the application to the investigation of sampled-data control systems of a special mathematical apparatus, such as the discrete Laplace transformation, is not justified methodologically. In the present paper, therefore, the properties of sampled--data control systems are determined on the basis of the well-known and more general integral Laplace transformation. Owing to the application of the integral Laplace transformation and to the direct introduction of the definitions concerning the transfer function and the frequency characteristic of a sampled-data control system by the formulae

$$K^*(e^s, \varepsilon) \underset{\text{df}}{=} \frac{\mathcal{L}\{x_2[\bar{t}, \varepsilon]\}}{\mathcal{L}\{x_1[\bar{t}, 0]\}},$$

$$M^*(j\bar{\omega}, \varepsilon) \underset{\text{df}}{=} \left.\frac{\mathcal{L}\{x_2[\bar{t}, \varepsilon]\}}{\mathcal{L}\{x_1[\bar{t}, 0]\}}\right|_{s=j\bar{\omega}}$$

we achieve a greater simplicity of method and bring it closer to the well-known methods of investigating dynamical systems.

The present paper is concerned with the problem of synthesis of sampled-data control systems; the methods of analysis of sampled-data control systems have recently been considerably advanced, while research in the methods of synthesis have so far been almost entirely neglected.

The object of the paper is to investigate the basic properties of frequency and time characteristics of sampled-data control systems from the point of view of the possibility of utilizing those properties in the problems of synthesis. It should be emphasized that no well-known properties of the frequency characteristics of systems with continuous action should, in the domain of sampled-data control systems, be assumed by analogy as valid — unless corresponding proofs are supplied, since such a procedure would evidently be erroneous and inadmissible. This follows from the fact that, for example, the frequency characteristics of sampled--data control systems constitute notions more general in character than those of the frequency characteristics of systems with continuous action. Moreover, it is impossible to establish a corresponding isomorphism between the characteristics of the two types of systems. Consequently, all the conclusions stated in this paper are supported by appropriate proofs. Attention is directed to the fact that it is not always the case that the properties of the characteristics of sampled-data control systems established in this paper are compatible with the corresponding characteristics of systems with continuous action.

2. CHARACTERISTICS OF SAMPLED-DATA CONTROL SYSTEMS

After splitting a feedback path in a sampled-data control system we obtain a sampled-data system of the form shown in Fig. 3. This system consists of two elements, in a series connection — *a linear element* and *a sampler*.

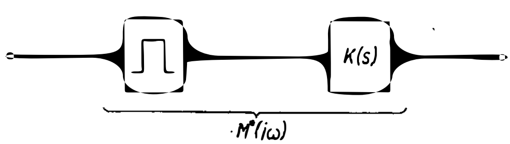

Fig. 3. Sampled-data system

It is well-known from the theory of dynamical systems that *the transmission properties* of a linear system can always be uniquely determined by means of the function $K(s)$ of the complex variable s

$$K(s) = \frac{\mathcal{L}\{x_2(t)\}}{\mathcal{L}\{x_1(t)\}}, \tag{1}$$

where x_2 and x_1 are, respectively, the input and the output signal of the system under consideration, and $\mathcal{L}$ denotes the integral Laplace transformation.

The question now arises as to whether it is possible in a similar manner to determine the capability of transmitting signals by a sampled--data dynamical system with the aid of a certain function. The answer is positive.

In order to explain this let us first observe that any continuous function of the real variable t can be uniquely determined by a set of step functions dependent on a certain real parameter. In fact, if we take into consideration, for example, the function $f(\bar{t})$, where $\bar{t} = \frac{t}{T}$ is a real variable, then the set of step functions $f[\bar{t}, \varepsilon]$ dependent on the parameter $\varepsilon \in [0, 1]$ and made up of the values of the function $f(\bar{t})$, uniquely determines the function $f(\bar{t})$ (Fig. 4). This can be expressed by the formula

$$f(\bar{t})[=]f[\bar{t}, \varepsilon]; \quad 0 \leqslant \varepsilon < 1. \tag{2}$$

The pulse transfer function, which determines the capability of the sample-data system of transmitting signals, is defined as

$$K^*(e^s, \varepsilon) \underset{\mathrm{df}}{=} \frac{\mathcal{L}\{x_2[t, \varepsilon]\}}{\mathcal{L}\{x_1[t]\}}; \quad 0 \leqslant \xi < 1, \tag{3}$$

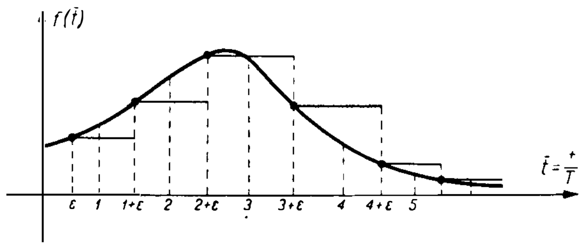

Fig. 4. Continuous function $f(t)$ determined by the set of step functions $f[\bar{t},\varepsilon]$

where $x_2[\bar{t}, \varepsilon]$ is a step function dependent on the real parameter ε, and determining the output signal $x_2(t)$ of the sampled-data system, $x_1(\bar{t})$ is a step function made up of the function $x_1(t)$ of the input signal of the sampled-data system.

It can be shown that depending on the value of the parameter ε, the function K^* may be expressed by two different analytical formulae

$$K^*_{\mathrm{I}}(e^s, \varepsilon) \text{ for the interval } 0 \leqslant \varepsilon < \gamma$$

and

$$K^*_{\mathrm{II}}(e^s, \varepsilon) \text{ for the interval } \gamma \leqslant \varepsilon < 1.$$

The function $K^*(e^s, \varepsilon)$ has a simple physical interpretation.

Now we have to prove that $K^*(e^s, \varepsilon)$ is the Laplace transformation of the function $h_y[\bar{t}, \varepsilon]$ divided by $\frac{1}{s}(1 - e^{-s})$, where $h_y[\bar{t}, \varepsilon]$ is the function of the output signal, when the linear part of the sampled-data system is actuated by an excitation in the form of a rectangular pulse having width γT and amplitude equal to unity (Fig. (5) [2]).

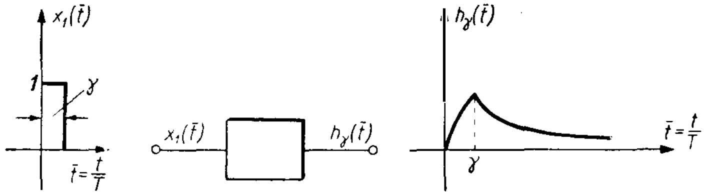

Fig. 5. Response of a system to an excitation in the form of a rectangular pulse

Thus we have

$$K^*(e^s, \varepsilon) = \frac{\mathfrak{L}\{h_\gamma[\bar{t}, \varepsilon]\}}{\eta}, \tag{4}$$

where

$$\eta = \frac{1}{s}(1 - e^{-s}).$$

We call frequency characteristic of a sampled-data system the diagram of the complex function

$$M^*(j\bar{\omega}, \varepsilon) \underset{\text{df}}{=} \left. \frac{\mathfrak{L}\{x_2[\bar{t}, \varepsilon]\}}{\mathfrak{L}\{x_1[\bar{t}]\}} \right|_{s=j\bar{\omega}}; \quad \bar{\omega} = \omega T \tag{5}$$

$\bar{\omega} = \omega T$ is called the dimensionless angular frequency.

An interesting property of the characteristic $M^*(j\bar{\omega}, \varepsilon)$ is its dependence on the real parameter ε. Each value of the parameter ε corresponds in general to a different frequency characteristic.

In view of the above definition we have

$$M^*(j\bar{\omega}, \varepsilon) = K^*(e^{j\bar{\omega}}, \varepsilon), \tag{6}$$

— that is, the frequency characteristic of a sampled-data system is obtained directly from the pulse transfer function K^* by substituting the variable $j\bar{\omega}$. for the complex variable s.

It can be proved that the frequency pattern determined by Eq. (5) in the case of a stable sampled-data system satisfies the relation

$$M^*(j\bar{\omega}, \varepsilon) = \frac{x_{2\text{steady}}[\bar{t}, \varepsilon]}{x_1[\bar{t}]}, \tag{7}$$

---

[2] It is assumed that the sampler and the linear element are what are called "isolated" systems.

in which $x_{2\,\mathrm{steady}}[\bar{t}, \varepsilon]$ is a steady-state component of the output signal $x_2[\bar{t}, \varepsilon]$, actuated by the harmonic excitation $x_1[\bar{t}] = e^{j\bar{\omega}n}$.

Note that Tsypkin [1] determines the frequency characteristic of a sampled-data system by Formula (7). The pattern so determined *is no longer valid in the case of astatical sampled-data systems* which are in common use in sampled-data equipments.

The frequency characteristic of a sampled-data system, as a complex function, may be represented in the form of the sum of its components

$$M^*(j\bar{\omega}, \varepsilon) = A^*(\bar{\omega}, \varepsilon) + j\,B^*(\bar{\omega}, \varepsilon) \tag{8}$$

or in the exponential form

$$M^*(j\bar{\omega}, \varepsilon) = M_0^*(\bar{\omega}, \varepsilon)\, e^{j\varphi^*}(\bar{\omega}, \varepsilon). \tag{9}$$

The functions A^* and B^* are called a real and an imaginary component of the frequency characteristic, and M_0^* and φ^*, respectively, are referred to as the amplitude characteristic and the phase characteristic.

A time characteristic of a sampled-data system is called the plot of the function of the output signal actuated by an excitation by means of a unit function (Fig. 6).

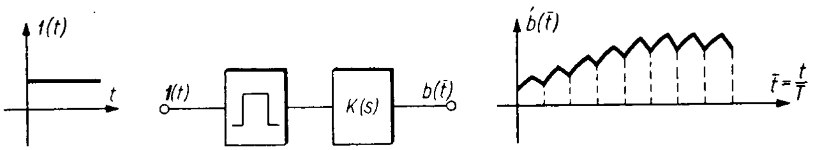

Fig. 6. Explanation for the time characteristic of a sampled-data system

3. PROPERTIES OF SAMPLED-DATA SYSTEM CHARACTERISTICS

In the paper by Tsypkin [1] the following relation is given between the frequency and the time characteristic of a sampled-data system

$$\Delta b[n-1, \varepsilon] = \frac{2}{\pi} \int_0^{\pi} A^*(\bar{\omega}, \varepsilon) \cos \bar{\omega}\, n \cdot d\bar{\omega}; \qquad n \geqslant 0. \tag{10}$$

This formula determines the relation between the functions b and A^* in a complicated form, and is therefore inconvenient in applications. Accordingly, Eq. (10) will be transformed so that it shall be reduced to a form more suitable for practical use.

If, in Formula (10) we take into consideration the obvious equality

$$\Delta\left\{\frac{\sin\bar{\omega}\left(n+\frac{1}{2}\right)}{\sin\frac{\bar{\omega}}{2}}\right\}=2\cos\bar{\omega}(n+1),$$

then we obtain

$$\Delta b[n,\varepsilon]=\frac{1}{\pi}\int_0^{\pi}A^*(\bar{\omega},\varepsilon)\,\Delta\left\{\frac{\sin\bar{\omega}\left(n+\frac{1}{2}\right)}{\sin\frac{\bar{\omega}}{2}}\right\}d\bar{\omega},$$

and hence

$$b[n,\varepsilon]=\frac{1}{\pi}\int_0^{\pi}A^*(\bar{\omega},\varepsilon)\frac{\sin\bar{\omega}\left(n+\frac{1}{2}\right)}{\sin\frac{\bar{\omega}}{2}}d\bar{\omega}+\lambda.$$

Since for $A^* = 0$ there must be $b[n, \varepsilon] = 0$, then $\lambda = 0$ and

$$\boxed{b[n,\varepsilon]=\frac{1}{\pi}\int_0^{\pi}A^*(\bar{\omega},\varepsilon)\frac{\sin\bar{\omega}\left(n+\frac{1}{2}\right)}{\sin\frac{\bar{\omega}}{2}}d\bar{\omega};\quad n\geqslant 0} \tag{11}$$

This formula corresponds to the well-known formula

$$h(t)=\frac{1}{\pi}\int_0^{\infty}P(\omega)\frac{\sin\omega t}{\omega}d\omega;\qquad t\geqslant 0$$

determining the relation between the real component $P(\omega)$ of the frequency characteristic and the time characteristic $h(t)$ in the case of systems with continuous action.

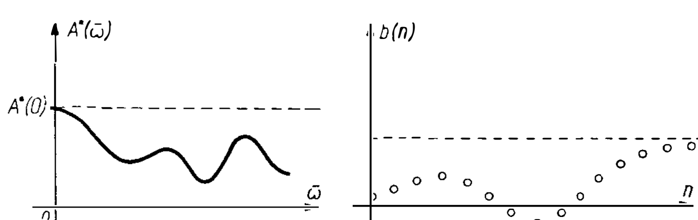

Fig. 7. Characteristics $A^*(\bar{\omega})$ and $b(n)$ satisfying the Property 1

For engineering applications, it is very important to investigate whether certain properties of the function $b[n, \varepsilon]$ which are of interest to us can be determined from the character of the plot of the function $A^*(\bar{\omega}, \varepsilon)$. To this end, we shall now prove certain properties specified below.

Property 1

If in the interval $\bar{\omega} \in [0, \pi]$, $A^*(\bar{\omega}\ 0) \geqslant 0$ and in addition $A^*(0, 0) \geqslant A^*(\bar{\omega}, 0)$ (Fig. 7), then the inequality

$$b(0) \leqslant b(\infty)^{3)} \tag{12}$$

holds.

Proof. In fact, from Formula (11) we have

$$b(0) = \frac{1}{\pi} \int_0^\pi A^*(\bar{\omega})\, d\bar{\omega}.$$

If then we assume that $A^*(\bar{\omega}) \geqslant 0$ and $A^*(\bar{\omega}) \leqslant A^*(0)$ for $\bar{\omega} \in [0, \pi]$, we shall obtain

$$b(0) \leqslant \frac{1}{\pi} \int_0^\pi A^*(0)\, d\bar{\omega} = \frac{1}{\pi} A^*(0)\, \bar{\omega} \Big|_0^\pi = A^*(0).$$

However, since

$$A^*(0) = b(\infty)$$

then we shall have

$$b(0) \leqslant b(\infty)$$

quod erat demonstrandum.

Property 2

If $A^*(\bar{\omega})$ in the interval $\bar{\omega} \in [0, \pi]$ is a positive function monotonically decreasing (Fig. 8), then the inequality

$$b(n) \leqslant 1.22 b(\infty) \tag{13}$$

holds.

Proof. Let us express the integral (11) as the sum of integrals

$$\int_0^\pi A^*(\bar{\omega}) \frac{\sin \frac{\bar{\omega}}{2}(2n+1)}{\sin \frac{\bar{\omega}}{2}}\, d\bar{\omega} =$$

---

[3]) For simplicity, the functions $A^*(\bar{\omega}, 0)$ and $b[n, 0]$ will further be denoted by $A^*(\bar{\omega})$ and $b(n)$, respectively.

$$= \sum_{\nu=0}^{n-1} \int_{\nu\frac{2\pi}{2n+1}}^{(\nu+1)\frac{2\pi}{2n+1}} A^*(\bar{\omega}) \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}} d\bar{\omega} +$$

$$+ \int_{n\frac{2\pi}{2n+1}}^{\pi} A^*(\bar{\omega}) \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}} d\bar{\omega} = a_0 - a_1 + a_2 - \ldots +$$

$$+ (-1)^{n-1} a_{n-1} + (-1)^n a_R; \qquad n = 0, 1, 2, \ldots,$$

where $a_\nu > 0$; $\nu = 0, 1, 2, \ldots\ldots$

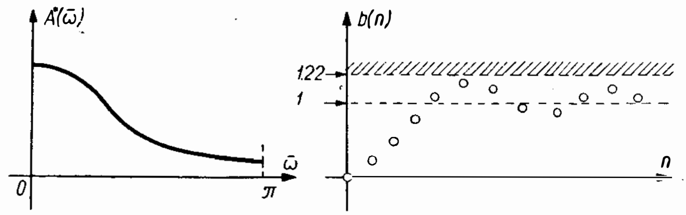

Fig. 8. Characteristics satisfying the Property 2

Since in the interval $\bar{\omega} \in [0, \pi]$ the function $\frac{1}{\sin\frac{\bar{\omega}}{2}}$ is a positive function monotonically decreasing, and by the assumption $A^*(\bar{\omega})$ is also a positive function monotonically decreasing, we have

$$a_0 > a_1 > a_2 \ldots > a_{n-1} > a_R.$$

Hence we conclude that

$$\int_0^{\pi} A^*(\bar{\omega}) \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}} d\bar{\omega} \leqslant \int_0^{\frac{2\pi}{2n+1}} A^*(\bar{\omega}) \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}} d\bar{\omega} \leqslant$$

$$\leqslant A^*(0) \int_0^{\frac{2\pi}{2n+1}} \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}} d\bar{\omega}.$$

Thus

$$b(n) \leqslant b(\infty) \cdot \varphi(n),$$

where

$$\varphi(n) = \frac{1}{\pi} \int_0^{\frac{2\pi}{2n+1}} \frac{\sin \frac{\bar{\omega}}{2} (2n + 1)}{\sin \frac{\bar{\omega}}{2}} d\bar{\omega}.$$

It can easily be verified that the function $\varphi(n)$ is a positive function monotonically decreasing. Moreover, we have

$$\varphi(0) = 2$$

and

$$\varphi(1) = \frac{1}{\pi} \int_0^{\frac{2}{3}\pi} \frac{\sin \frac{2}{3} \bar{\omega}}{\sin \frac{\bar{\omega}}{2}} d\bar{\omega} = 1.2179 \ldots\ldots \approx 1.22.$$

Since it follows from the Property 1 that the inequality

$$b(0) \leqslant b(\infty),$$

must also be satisfied then

$$b(n) \leqslant 1.22 b(\infty)$$

quod erat demonstradum.

Property 3

If $A^*(\bar{\omega})$ is in the interval $\bar{\omega} \in [0, \pi]$ a monotonically decreasing function (not entirely positive) (Fig. 9), then the following inequality holds:

$$\frac{b(n)}{b(\infty)} \leqslant 1.22 - 0.22 \frac{A^*(\pi)}{A^*(0)}. \tag{14}$$

Proof. Let us discuss the formula

$$b(n) = \frac{1}{\pi} \int_0^{\pi} A^*(\bar{\omega}) \frac{\sin \frac{\bar{\omega}}{2} (2n + 1)}{\sin \frac{\bar{\omega}}{2}} d\bar{\omega}.$$

The function $A^*(\bar{\omega})$ can always be represented as the sum of two functions $A_1^*(\bar{\omega})$ and $A^*(\pi)$,

$$A^*(\bar{\omega}) = A_1^*(\bar{\omega}) + A^*(\pi),$$

where $A_1^*(\bar{\omega})$ is in the interval $\bar{\omega} \in [0, \pi]$ a positive function monotonically decreasing. Therefore

$$b(n) = \frac{1}{\pi} \int_0^{\pi} [A_1^*(\bar{\omega}) + A^*(\pi)] \frac{\sin \frac{\bar{\omega}}{2} (2n + 1)}{\sin \frac{\bar{\omega}}{2}} d\bar{\omega} =$$

$$= \frac{1}{\pi}\int_0^\pi A_1^*(\bar{\omega})\frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}}\,d\bar{\omega} + A^*(\pi)$$

and, since $A_1^*(\bar{\omega})$ is a positive function monotonically decreasing

$$b(n) \leqslant 1.22A_1^*(0) + A^*(\pi) = 1.22[A^*(0) - A^*(\pi)] + A^*(\pi) = \\ = 1.22A^*(0) - 0.22\,A^*(\pi).$$

Hence

$$\frac{b(n)}{b(\infty)} \leqslant 1.22 - 0.22\frac{A^*(\pi)}{A^*(0)}$$

quod erat demonstrandum.

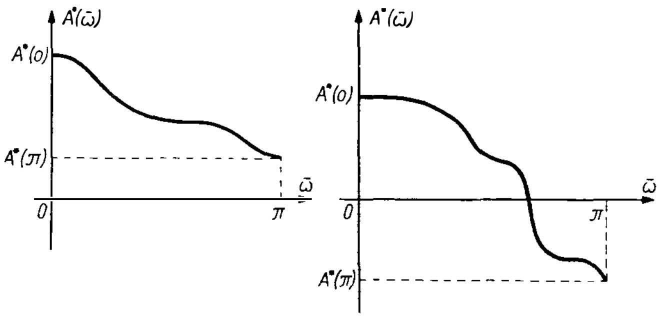

Fig. 9. Characteristics satisfying the Property 3

For estimating the overshoot σ we thus have the following formula

$$\sigma\% = \frac{b(n) - b(\infty)}{b(\infty)}\,100\% \leqslant \left(0{,}22 - 0{,}22\frac{A^*(\pi)}{A^*(0)}\right)100\%.$$

If for instance $\frac{A^*(\pi)}{A(0)} = 0.3$, then $\sigma \leqslant 16\%$, and if $\frac{A^*(\pi)}{A^*(0)} = 0.3$, then $\sigma \leqslant 29\%$.

Property 4

If $A^*(\bar{\omega})$ in the interval $\bar{\omega} e[0, \bar{\omega}_1]$ is a positive function monotonically increasing, whereas in the interval $\bar{\omega}\, e[\bar{\omega}_1\, \pi]$ — it is a positive function monotonically decreasing (Fig. 10), then we have

$$\frac{b(n)}{b(\infty)} < 1.22\frac{A^*_{max}}{A^*(0)}\,. \tag{15}$$

Proof. The frequency pattern $A^*(\bar{\omega})$ can be represented as the difference of the two monotonically decreasing patterns (Fig. 10)

$$A^*(\bar{\omega}) = A_1^*(\bar{\omega}) - A_2^*(\bar{\omega}),$$

where $A_1^*(0) = A^*_{\max}$. Therefore

$$b(n) = \frac{1}{\pi}\int_0^{\pi} A_1^*(\bar\omega)\frac{\sin\frac{\bar\omega}{2}(2n+1)}{\sin\frac{\bar\omega}{2}}\,d\bar\omega - \frac{1}{\pi}\int_0^{\pi} A_2^*(\bar\omega)\frac{\sin\frac{\bar\omega}{2}(2n+1)}{\sin\frac{\bar\omega}{2}}\,d\bar\omega,$$

and, since $A_2^*(\bar\omega) \geqslant 0$,

$$b(n) < \frac{1}{\pi}\int_0^{\pi} A_1^*(\bar\omega)\frac{\sin\frac{\bar\omega}{2}(2n+1)}{\sin\frac{\bar\omega}{2}}\,d\bar\omega.$$

Hence by virtue of the property 2

$$b(n) < 1.22^{*}_{\max}$$

and finally

$$\frac{b(n)}{b(\infty)} < 1.22\,\frac{A^*_{\max}}{A^*(0)}$$

quod erat demonstrandum.

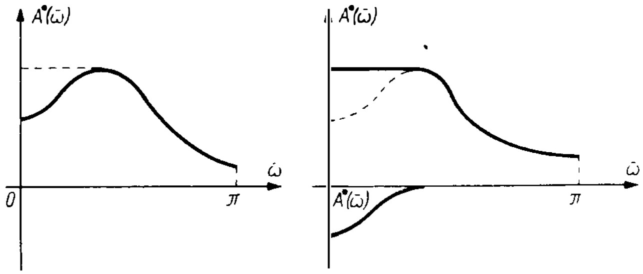

Fig. 10. Characteristic satisfying the Property 4

Hence the following conclusion may be arrived at: if the frequency pattern $A^*(\bar\omega)$ has a maximum equal to $A^*_{\max}$, then the overshoot σ has a smaller value than $1.22\,\frac{A^*_{\max}}{A^*(0)} - 1$ and, consequently, the following relation holds

$$\sigma\% < \left(1.22\,\frac{A^*_{\max}}{A^*(0)} - 1\right)\cdot 100\%.$$

If, for example

$$\frac{A^*_{\max}}{A^*(0)} = 1.25,$$

then the overshoot cannot exceed 53% (in this case, for systems with continuous control we should have $\sigma\% < 48\%$).

Property 5

If $A^*(\bar{\omega})$ in the interval $\omega\,\varepsilon[o, \pi]$ has a plot containing one maximum, as indicated at Fig. 11 (and is a function not necessarily positive), then we have

$$\frac{b(n)}{b(\infty)} < \frac{1.22A^*_{\max} - 0.22A^*(\pi)}{A^*(0)} \tag{16}$$

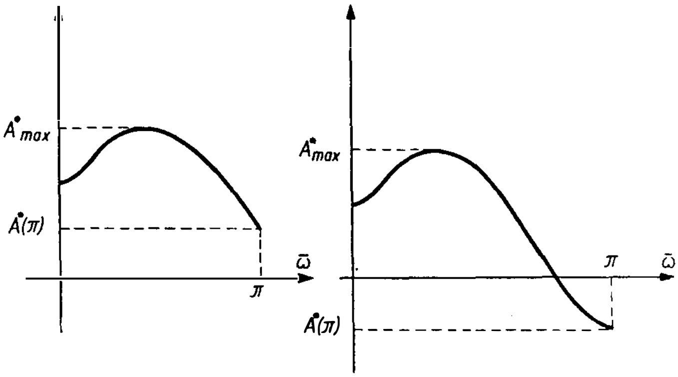

Fig. 11. Characteristic satisfying the Property 5

Proof. The function $A_1^*(\bar{\omega})$, which was discussed in connection with the Property 4, can be represented as the sum of two functions $A^*(\bar{\omega})$ and $A^*(\bar{\omega})$ in a manner identical with that in the proof for the Property 3. Accordingly, in accordance with Formula (59) we shall write

$$b(n) < \frac{1}{\pi}\int_0^{\pi} [A_2^*(\bar{\omega}) + A^*(\pi)] \frac{\sin\frac{\bar{\omega}}{2}(2n+1)}{\sin\frac{\bar{\omega}}{2}}\, d\bar{\omega} \leqslant 1.22A_2^*(0) + A^*(\pi) =$$

$$= 1.22\,[A_1^*(0) - A^*(\pi)] + A^*(\pi) = 1.22A_1^*(0) - 0.22A^*(\pi).$$

Now, taking into consideration that

$$A_1^*(0) = A^*_{\max}$$

we shall finally obtain

$$b(n) < 1.22A^*_{\max} - 0.22A^*(\pi)$$

and

$$\frac{b(n)}{b(\infty)} < \frac{1.22A^*_{\max} - 0.22A^*(\pi)}{A^*(o)}, \text{ quod erat demonstradum.}$$

The overshoot for the pattern under consideration may therefort be estihated in terms of the following equation

$$\partial\,\% < \left(1.22\frac{A^*_{\max}}{A^*(o)} - 0.22\frac{A^*(\pi)}{A^*(o)} - 1\right)100\,\% \qquad (17)$$

Property 6

If $A^*(\bar{\omega})$ is a positive function monotonically decreasing, then the following inequality holds

$$\Delta b\,(n-1) \leqslant \frac{2}{\pi}\,\frac{b\,(\infty)}{n}. \qquad (18)$$

Proof. Let us take into consideration the formula

$$\Delta b\,(n-1) = \frac{2}{\pi}\int_0^\pi A^*\,\bar{\omega}\cos n\bar{\omega}d\bar{\omega}.$$

Expanding the integral appearing on the right-hand side of the above formula into the sum of corresponding integrals, we obtain

$$\Delta b\,(n-1) = \mathrm{e} + \sum_{k=0}^{n-2}(-1)^{k+1}a^k + (-1)^{n-1}a_{n-1},$$

where

$$\mathrm{e} = \frac{2}{\pi}\int_0^{\frac{\pi}{2n}} A^*(\bar{\omega})\cos n\bar{\omega}d\bar{\omega},$$

$$a_k = \left|\frac{2}{\pi}\int_{\frac{2k+1}{2n}\pi}^{\frac{2k+3}{2n}\pi} A^*(\bar{\omega})\cos n\bar{\omega}d\bar{\omega}\right|; \qquad k = 0, 1, 2, \ldots, n-2,$$

$$a_{n-1} = \left|\frac{2}{\pi}\int_{\frac{2n-1}{2n}\pi}^{\pi} A^*(\bar{\omega})\cos n\bar{\omega}d\bar{\omega}\right|.$$

If $A^*(\bar{\omega})$ is a positive function monotonically decreasing, we have the inequalities

$$a_0 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-2} \geqslant a_{n-1} > 0.$$

Accordingly

$$\Delta b(n-1) \leqslant e = \frac{2}{\pi} \int_0^{\frac{\pi}{2n}} A^*(\bar{\omega}) \cos n\bar{\omega} d\bar{\omega}. \tag{19}$$

Now we may notice that owing to the inequality $A^*(\bar{\omega}) \leqslant A^*(0)$

$$\int_0^{\frac{\pi}{2n}} A^*(\bar{\omega}) \cos n\bar{\omega} d\bar{\omega} \leqslant A^*(0) \int_0^{\frac{\pi}{2n}} \cos n\bar{\omega} d\bar{\omega} =$$

$$= A^*(0) \left[\frac{1}{n} \sin n\bar{\omega}\right]_0^{\frac{\pi}{2n}} = A^*(0) \frac{1}{n}.$$

Therefore by virtue of Formula (19), and owing to the inequality $A^*(0) = b(\infty)$, we obtain

$$\Delta b(n-1) \leqslant \frac{2}{\pi} \frac{b(\infty)}{n} \text{quod erat demonstrandum.}$$

The above formula may also be written as

$$\frac{\Delta b(n-1)}{b(\infty)} \leqslant 0.64 \frac{1}{n}. \tag{19a}$$

Property 7

If $A^*(\bar{\omega})$is in the interval $\bar{\omega}\epsilon[0, \pi]$ a positive function possessing a negative and monotonically increasing derivative (Fig. 12), then the time characteristic $b(n)$ is a monotonically increasing function, and hence

$$\Delta b(n) \geqslant 0. \tag{20}$$

Proof. Let us consider the Formula

$$\Delta b(n-1) = \frac{2}{\pi} \int_0^{\pi} A^*(\bar{\omega}) \cos n\bar{\omega} d\bar{\omega}.$$

The characteristic $b(n)$ is a monotonically increasing function if

$$\Delta b(n-1) \geqslant 0; \quad n = 1, 2, \ldots$$

The frequency characteristic will be reperesented as the sum ot the characteristics $A_k^*(\bar{\omega})$ having the shape of triangles

$$A_k^*(\bar{\omega}) = \begin{cases} A_k^*(0)\left(1 - \dfrac{\bar{\omega}}{\bar{\omega}_k}\right); & 0 \leqslant \bar{\omega} < \bar{\omega}_k \\ 0; & \bar{\omega} \geqslant \bar{\omega}_k, \end{cases}$$

where

$$A_k^*(0) > 0.$$

When the number of *triangular characteristics* tends to infinity, the sum thereof in the limit will represent the function $A^*(\bar{\omega})$.
Hence

$$A^*(\bar{\omega}) \approx \sum_{k=0}^{n} A_k(\bar{\omega})$$

where

$$0 < \omega_0 < \omega_1 < \ldots < \omega_n .$$

Introducing now the following notation

$$\Delta b_k (n-1) - \frac{2}{\pi} \int A_k^*(0) \left(1 - \frac{\bar{\omega}}{\omega_k}\right) \cos n\bar{\omega} d\bar{\omega} ,$$

we shall obtain

$$\Delta b_k (n-1) = \frac{2A_k^*(0)}{\omega_k \pi n^2} (1 - \cos n\omega_k)$$

and

$$\Delta b_k (n-1) \geqslant 0 ,$$

where an equality holds for $n = \frac{2\pi}{\omega_k} k$ quod erat demonstrandum.

If the frequency characteristic $A^*(\bar{\omega})$ satisfies the conditions specified in the section referring to the Property 7, then the sampled-data control

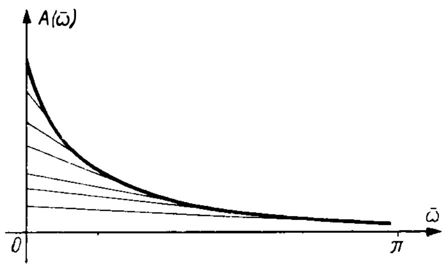

Fig. 12. Characteristic satisfying the Property 6

system has a monotonical characteristic, that is to say, with no overshoot and no oscillations.

Now we shall prove that there exist other properties in terms of which we can determine certain features of the time characteristic directly from the shape of the frequency characteristic of the linear element of the system.

Property 8

If the real component $P(\omega)$ of the frequency pattern $K(j\omega)$ of the linear element of a sampled-data system possesses a negative derivative monotonically incereasing for $\omega \geqslant 0$ (Fig 13), then, with the assumption of very narrow pulses in relation to the period of generating pulses ($\gamma << 1$), the characteristic $A^*(\bar{\omega})$ in the interval $\bar{\omega} \in [0, \pi]$ is a monotonically decreasing function.

Proof. According to Tsypkin, the relation between the frequency characteristic $M^*(\bar{\omega})$ of a sampled-data system for $\gamma \ll 1$ and the time characteristic of its linear element (with the assumption that $\lim_{S \to \infty} s K(s) = 0$), is defined by the formula

$$M^*(j\bar{\omega}) = \bar{k} \sum_{m=-\infty}^{\infty} K[j(\bar{\omega} + 2\pi m)],$$

where $\bar{k}$ is a constant dependent on the design of the sampler. Since M^* and K are complex functions, and

$$M^*(j\bar{\omega}) = A^*(\bar{\omega}) + jB^*(\bar{\omega}),$$
$$K(j\omega) = P(\omega) + jQ(\omega),$$

then

$$A^*(\bar{\omega}) = \bar{k} \sum_{m=-\infty}^{\infty}{}' P(\bar{\omega} + 2\pi m).$$

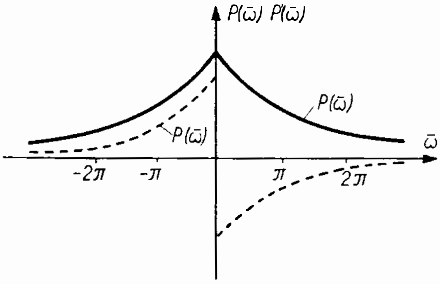

Fig. 13. Characteristic which ensures the monotonical function $A^*(\bar{\omega})$

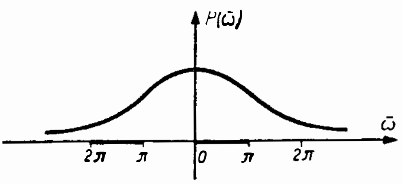

Fig. 14. Characteristic which does not ensure the monotonical function $A^*(\bar{\omega})$

This series with the above assumptions is a strong-convergent and differentiable series; we may thus write

$$A^{*\prime}(\bar{\omega}) = \bar{k}[P'(\bar{\omega}) + P'(\bar{\omega} - 2\pi) + P'(\bar{\omega} + 2\pi) +$$
$$+ P'(\bar{\omega} + 4\pi) + P'(\bar{\omega} - 4\pi) + \ldots]$$

Since we assume that $P(\omega)$ is a monotonically decreasing function — that is,

$$P'(\bar{\omega}) \leqslant 0,$$

then — taking into consideration that the function $P(\bar{\omega})$ is even for the interval $\bar{\omega} \in [0, \pi]$ — we shall obtain the inequalities

$$P'(\bar{\omega}) + P'(\bar{\omega} - 2\pi) \leqslant 0,$$
$$P'(\bar{\omega} + 2\pi) + P'(\bar{\omega} - 4\pi) \leqslant 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P'[\bar{\omega} + 2\pi(N-1)] - P'(\bar{\omega} - 2\pi N) \leqslant 0.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since these inequalities hold for any natural N, then we have

$$A^*(\bar{\omega}) \leqslant 0 \text{ for } \bar{\omega} \in [0, \pi] \tag{21}$$

which means that the real component of the frequency characteristic of a sampled-data system is in the interval $\bar{\omega} \in [0, \pi]$ and with assumption that $\gamma << 1$, it is a monotonically decreasing function, which was to be proved.

It should be emphasized that the condition requiring that the function $P(\omega)$ be monotonical is not sufficient for the function $A^*(\bar{\omega})$ to be monotonical in the interval $\bar{\omega} \in [0, \pi]$ (Fig. 14).

Note that the following relations also hold

$$A^*(0) = \bar{k}\,[P(0) + 2 \sum_{k=0}^{\infty} P(2k\pi)],$$

$$A^*(\pi) = 2\bar{k} \sum_{k=0}^{\infty} P\,[(2k+1)\pi].$$

If $P(\bar{\omega})$ decreases rapidly with the increase in frequency, we then have

$$\left.\begin{aligned} A^*(0) &\approx \bar{k}\,[P(0) + 2P(2\pi) + 2P(4\pi)], \\ A^*(\pi) &\approx 2\bar{k}\,[P(\pi) + P(3\pi) + P(5\pi)]. \end{aligned}\right\} \tag{22}$$

Making use of Formulae (22) on the basis of the Formula

$$\frac{b(n)}{b(\infty)} \leqslant 1.22 - 0.22 \frac{A^*(\pi)}{A^*(0)}$$

we can determine the *upper limit* of the waveform or the time characteristic $b(n)$ directly from our knowledge of the plotted function $P(\omega)$, since we have the relation

$$\frac{b(n)}{b(\infty)} \leqslant 1.22 - 0.44 \frac{P(\pi) + P(3\pi) + P(5\pi)}{P(0) + 2P(2\pi) + 2P(4\pi)}$$

This relation holds in the case in which $P'(\bar{\omega}) \leqslant 0$ for $\omega \geqslant 0$ that is, when $P(\bar{\omega})$ is a strictly monotonical function decreasing for positive frequencies $\omega \geqslant 0$.

Property 9

If the unit-step-function characteristic $h(t)$ of a linear element is a monotonically increasing function, the unit-step-function characteristic $b(n)$ of the sampled-data system is also a monotonically increasing sequence (Fig. 15).

Proof. Between the characteristic $b(n)$ of a sampled-data system and the time characteristic $h(t)$ of the linear element the relation

$$\Delta b(n-1) = \bar{k} h_\gamma(n); \quad n \geqslant 1$$

holds, where

$$h_\gamma(n) = h(n) - h(n-\gamma).$$

If $h(t)$ is a monotonically increasing function, then

$$h(n) \geqslant h(n-\gamma)$$

for each $\gamma \in [0, 1]$. Therefore

$$h_\gamma(n) \geqslant 0$$

and

$$\Delta b(n-1) \geqslant 0; \quad n \geqslant 1, \tag{23}$$

and hence it follows that $b(n)$ in the interval $n \geqslant 0$ is a monotonically increasing function. Quod erat demonstrandum.

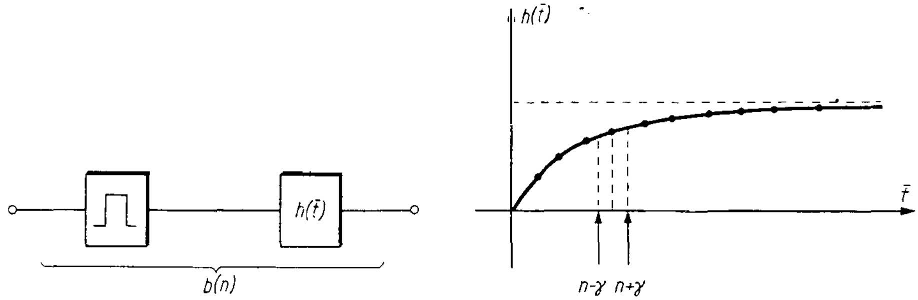

Fig. 15. Time characteristic $h(t)$ of the linear element of a sampled-data system

Property 10

If in the interval $\omega \in [0, \infty]$ the real component of the frequency characteristic of a sampled-data linear element satisfies the conditions

$$P(\omega) \geqslant 0; \quad \frac{dP(\omega)}{d\omega} \leqslant 0; \quad \frac{d^2 P(\omega)}{d^2 \omega} \geqslant 0, \tag{24}$$

then the time characteristic $b(n)$ of the sampled-data system is a monotonically increasing function.

Proof. The Property 10 follows immediately from the Property 9, since the time characteristic $h(t)$ is a monotonically increasing function in the case in which the conditions (24) are satisfield. Quod erat demonstrandum.

4. RELATION BETWEEN THE NYQUIST DIAGRAM OF THE OPEN-LOOP SYSTEM AND THE QUALITY OF SAMPLED-DATA CONTROL

The frequency characteristic of a sampled-data control system is related with the frequency characteristic of an open-loop system by the formula

$$M_z^*(j\bar{\omega}) = \frac{K^*(e^{j\bar{\omega}}, 0)}{1 + K^*(e^{j\omega}, 0)} = \frac{M^*(j\bar{\omega})}{1 + M^*(j\bar{\omega})}. \tag{25}$$

This formula has a form identical with that of the analogous formula concerning the system with continuous action. Therefore, in order to determine $M_z^* = A_z^* + jB_z^*$ from our knowledge of $M^*(j\bar{\omega})$, we can make use of the existing diagrams and monograms worked out for system with continuous action (Fig. 19).

The following conclusions can thus be arrived at:

Conclusion 1.

The overshoot in a sampled-data control system does not exceed 22 percent, if the Nyquist diagram of the open-loop system does not enter the region limited on its left side by a straight line parallel to the imaginary axis and passing through the point $(-1 + j0)$, and by a circle of radius $p = \frac{1}{2}$ and having its centre at the point $\left(-\frac{1}{2} + j0\right)$, and if the Nyquist diagram has only one intersection point with each curve $A_z^* = \text{const}$.

Conclusion 2.

The time characteristic $b(n)$ of the closed-loop system is a monotonical function (therefore no overshoot occurs) if the Nyquist diagram does not enter the region limited on its left side by a straight line parallel to the imaginery axis and passing through the point $(-1 + j0)$ and by a circle of radius $p = \frac{1}{2}$ and having its centre at the point $\left(-\frac{1}{2}, j0\right)$ and if the Nyquist diagram has only one intersection point with each curve $A_z^* = \text{const}$; if, moreover the increments of the frequency $\Delta\omega$ corresponding to the interesection points with the parallel curves $A_z^* = \text{const}$, constitute a decreasing sequence with the growth of $\bar{\omega}$.

5. SUMMARY OF THE PROCEDURES ESSENTIAL FOR PERFORMING THE FREQUENCY SYNTHESIS OF A SAMPLED-DATA CONTROL SYSTEM

A diagram of a simple sampled-data control system is shown in Fig. 16. In order to carry out the synthesis of such a system in terms of the frequency-characteristic method, the engineer must perform a number of actions as listed below:

1. The characteristic $K_\Omega W_\Omega(j\omega)\frac{1}{sT_1}$ of the controled system and of the servomotor (Fig. 17) must to be plotted.

2. Graphically, by way of summing up the values of the function $K_\Omega W_\Omega[j(\bar{\omega} + 2\pi m)] \frac{1}{jT_1(\bar{\omega} + 2\pi m)}$ in terms of the formula used in describing the Property 8, the pulse characteristic $M^*(j\bar{\omega})$ must be determined (Fig. 18).

3. By applying to Fig. 19 a tracing paper with circles indicating the constant values of the real component $A_z(\bar{\omega})$ in the closed-loop system, the plot of $A_z^*(\bar{\omega})$ must be determined.

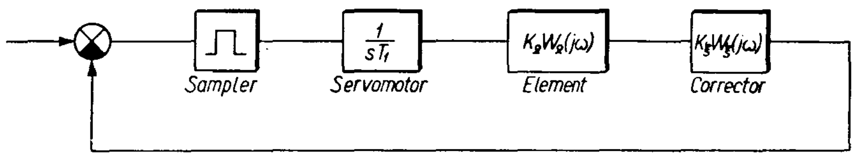

Fig. 16. Sampled-data control system

4. Making use of the properties just proved it may be verified whether the assumed overshoot σ is obtained. If σ is too high, then it must be considered how to correct the plot of $A_z^*(\bar{\omega})$ so that the overshoot shall be decreased.

5. It is also not difficult to realize what should be the plot of the (continuous) frequency characteristic so that a more suitable σ [4] will be obtained.

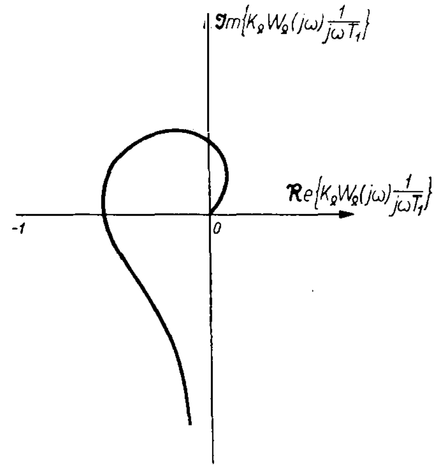

Fig. 17. Characteristic K

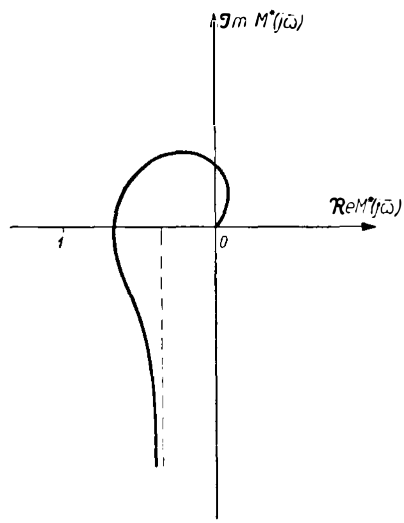

Fig. 18. Characteristic $M^*(j\bar{\omega}$

[4] It is possible to calculate and draw in advance several typical characteristics $KW(j\omega)$ and $M^*(\bar{\omega})$ corresponding to one another. The engineer — having at his disposal such characteristics — would have a better orientation how to modify the function $KW(j\omega)$ so that a more advantageous plot of the function $M^*(\bar{\omega})$ might be obtained.

6. Knowing the frequency characteristic of the controlled system and the required frequency characteristic of the open-loop system, the frequency characteristic of the corrector must be determined (by dividing the above characteristics one by another).

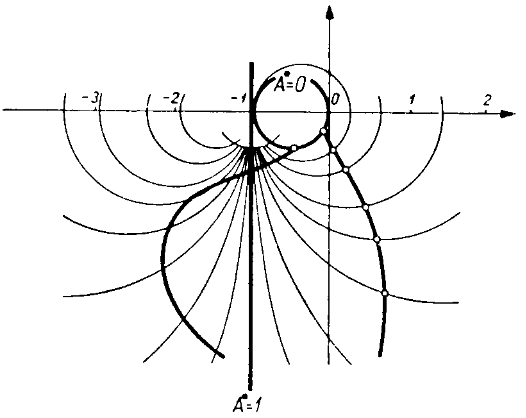

Fig. 19. Curves for the constant values of the real component of the frequency characteristic of a sampled-data control system

7. The synthesis of the corrector must be performed by means of well-known methods in terms of the given frequency characteristic.

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Translated by I. Bellert

TOPOLOGICAL CONSIDERATIONS AND SYNTHESIS OF LINEAR NETWORKS BY MEANS OF THE METHOD OF STRUCTURAL NUMBERS [1])

Two problems are discussed in the paper, namely: 1. formalization of the concept of a system, 2. analysis and synthesis of linear systems by means of the algebra of structural numbers.

As regards the first problem, the author has considered three different models of a system, namely, an abstract model, a topological model and a concrete (physical) model. A concept of the topological structure of a system and a concept of similar systems have been discussed along with other ones.

As regards the second problems, a new method is presented, namely, the so called algebra of structural numbers. It facilitates the analysis of linear electric systems and makes it possible to solve the problem of synthesis of a two-terminal or a four-terminal network in a general manner. By making use of a digital computer, we can find a set of systems with different structures which realize a given impedance function or a transfer function.

The algebra of structural numbers can also be applied, aside from electrical engineering, to solving linear algebraic equations. Moreover, it gives us a very simple algorithm for determination of all possible trees of a graph or a multigraph. The present paper is an extension of the method published by the author for the first time in the Journal of the Franklin Institute, December 1962 [1].

1. INTRODUCTION

There are several methods of analysis of linear electric systems. The traditional methods, which might also be called classical methods, for the most part make use of the theory of determinants and the matrix theory of Cayley. Such methods are quite sufficient for analyses of systems with not very complicated structure. At present, however, the degree of complication of systems e.g. in radio engineering, wire communication and automatic control, is often very high. The analysis of such systems by means of traditional methods brings about fairly cumbersome calculations. As a result, there appeared a class of new methods which are often called "topological methods". No really basic concepts of the algebraic nor combinatorial topology are involved in these methods. They are spoken of as topological, mainly because of the fact that the

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1) Archiwum Elektrotechniki (Vol. XII, No. 3, 1963, pp. 473—500).

configuration of the connections between the elements of the system makes possible the reading of their dynamical properties. As regards these methods, special credit should be given to the research of Chinese scientists initiated by Wang in 1934 [3]. I also wish to mention here the paper by H. Woźniacki [5] who contributed many new ideas to the concept of Wang.

The present paper is confined in principle to two problems, namely:

1) formalization of the concept of system,

2) analysis and synthesis of linear systems by means of a new method which the author proposes to call the algebra of structural numbers, and which is a continuation of the method of Wang.

The author believes that the algebra of structural numbers is a method on the basis of which it might be possible to solve the problems of the synthesis of linear systems in a general manner without introducing restrictions in relation to the structure of the system being designed.

2. FORMALIZATION OF THE CONCEPT OF SYSTEM

2.1. Introduction

It is worth noting that the concept of system is not usually specified precisely enough and is most often understood in an intuitive way. However, an intuitive understanding of the concept of system may bring about some misunderstandings, in particular when the analysis or synthesis of a system is to be carried out on the basis of the topological method. It is clear that in constructing a theory of a system, it is convenient to introduce such a theory in a most general manner, rather than to confine it to an electric system. Therefore, we are going to talk about the system conctived in terms of cybernetics in the hope of achieving thus greater generality.

We distinguish three basic concepts of a system, namely:

a) *an absract system* (based on the theory of sets),

b) *a topological system* constituting a geometrical representation of the abstract system,

c) *a concrete system* constituting a physical representation of the topological system.

The above idea is represented in Fig. 1.

By ascribing different contents to the abstract model, it is obviously possible to distinguish many different concrete systems, such as for instance, electric, hydraulic, mechanical, thermal, biological, economical, praxeological systems, and many others.

2.2. Abstract model of a system

The concept of an abstract system can be based on the *set* theory and the theory of ***relations***. Let us consider a finite subset M of a certain topological space (e.g. in the Euclidean space E^n) and determine in it a binary relation $\mathcal{R}$ as a subset of the Cartesian product $M \times M$

$$\mathcal{R} \subset M \times M \tag{1}$$

The relation $\mathcal{R}$ is then a set of *ordered pairs* $\langle x, y \rangle$ of certain elements of the set M.

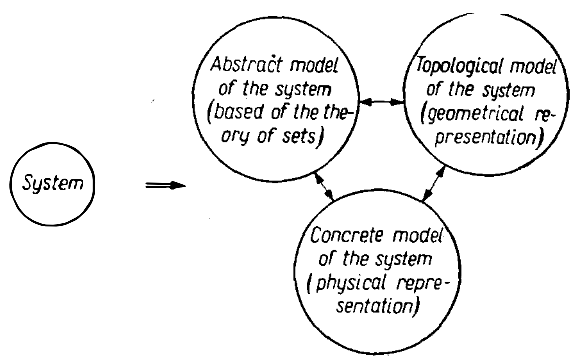


Fig. 1.

We write

$$x \mathcal{R} y \equiv \langle x, y \rangle \in \mathcal{R}. \tag{2}$$

The set of the antecedents of the ordered pairs belonging to $\mathcal{R}$ is called the left side of the domain

$$d_l \mathcal{R}.$$

The set of the consequents of these pairs is called the right side of the domain

$$d_r \mathcal{R}.$$

The sum of both domains of the relations is called the field of the relation and is denoted by $\mathcal{F}\mathcal{R}$; hence

$$\mathcal{F}\mathcal{R} = d_l \mathcal{R} + d_r \mathcal{R},$$

where the symbol $+$ is understood in the sense of the algebra of sets.

The elements of the left side domain $d_l \mathcal{R}$ are denoted by $d_l r$, and these of the right side domain $d_r \mathcal{R}$ by $d_r r$; hence

$$d_l r \in d_l \mathcal{R}; \quad d_r r \in d_r \mathcal{R}.$$

Now let us introduce the following definitions:

Definition 1. *A realation R will be called a structural relation in the set M,* *if and only if for each pair of non-empty sets X and Y satisfying the conditions*

$$X + Y = M, \quad X \cdot Y = 0, \tag{3}$$

there are elements $x \in X$ and $y \in Y$, for which at least one of the following relations holds

$$x \mathcal{R} y \text{ or } y \mathcal{R} x. \tag{4}$$

The notion of a structural relation will be used in determining an abstract system based on the theory of sets.

Defintion 2. *A relation $\mathcal{R}$ which is structural in the set of a topological space will be called a set-theory system. Elements of the relation $\mathcal{R}$ will be called the elements of a set-theory system.*

In virtue of the above definition the concept of a set-theory system is identified with the concept of a structural relation. From the definition of this relation it follows that the system contains no isolated elements in the sense that each element of the set M is in a determined relation $\mathcal{R}$ with at least one other element of the set M. The set M is then *the field* of the relation $\mathcal{R}$, that is

$$M = \mathcal{F} \mathcal{R}. \tag{5}$$

Elements of the field of the relation $\mathcal{R}$ will be called the vertices of the system.

If an element r is an ordered pair $\langle x, y \rangle$, we shall call x the beginning and y — the end of the element r, that is

$r = \langle x, y \rangle = x =$ the beginning, $y =$ the end of the element r.

Depending on the properties of the structural relation $\mathcal{R}$, we may distinguish several types of set-theory systems [2].

So, e.g. a set-theory system will be called the oriented system if the relation determining that system is asymmetric, i.e. it satisfies the condition

$$x \mathcal{R} y \Rightarrow y \text{ non } \mathcal{R} x, \text{ for any } x, y \in M. \tag{6}$$

If condition (6) is satisfied not for all the elements x, y of the set M, the system $\mathcal{R}$ will be called partly oriented.

A system will be spoken of as compact, if the relation determining it is a connected relation, i.e., it satisfies the condition

$$x, y \in M \Rightarrow x \mathcal{R} y \text{ or } y \mathcal{R} x. \tag{7}$$

If for certain $x \in M$ there holds the relation

$$x \mathcal{R} x$$

we say that the set-theory system possesses its own elements.

We shall now define the notions of input and output of a system.

Defintion 3. *The vertices of a system which are exclusively the beginnings of its elements will be called the inputs of the system; those which are merely the ends of its elements, will be called the outputs of the system.*

The inputs and outputs are called the boundary vertices of a system. The set of inputs and outputs is called the boundary of a system. All the vertices which do not belong to the boundary are called the interior of a system.

Let X denote the set of inputs, Y — the set of outputs, and B — the boundary of a system; we then obtain

$$X = d_l \mathcal{R} - d_r \mathcal{R}, \qquad Y = d_r \mathcal{R} - d_l \mathcal{R}, \qquad B = d_l \mathcal{R} \dot{-} d_r \mathcal{R}, \tag{8}$$

where the sign — denotes the symmetric difference of the sets $d_1 \mathcal{R}$ and $d_r \mathcal{R}$.

Set-theory systems may be broken down into the following categories:

1) initial systems — possessing no inputs, that is

$$d_l \mathcal{R} - d_r \mathcal{R} = 0, \tag{9}$$

2) final systems — possessing no outputs, that is

$$d_r \mathcal{R} - d_l \mathcal{R} = 0. \tag{10}$$

3) detached (isolated) systems — possessing no boundary

$$d_l \mathcal{R} \dot{-} d_r \mathcal{R} = 0, \tag{11}$$

4) relatively detached systems — possessing inputs and outputs, that is,

$$d_l \mathcal{R} - d_r \mathcal{R} \neq 0 \text{ and } d_r \mathcal{R} - d_l \mathcal{R} \neq 0. \tag{12}$$

A system consisting of a connection of an initial, a relatively detached and a final system is schematically presented in Fig. 2.

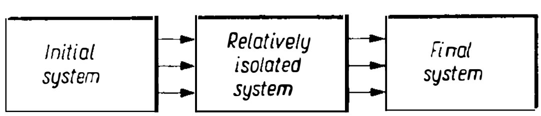


Fig. 2.

2.3. Topological model of a system

2.3.1. Definition of the topological system

The notion of *a one-dimensional simplex* will be understood as *a homeomorphic transformation* of a segment (*geometrical simplex*). Fig. 3

shows some examples of one-dimensional simplexes with the vertices x, y. The vertices of a simplex will be designated by circles. A one-dimensional simplex with the vertices x, y *will* be denoted by the symbol (x, y).

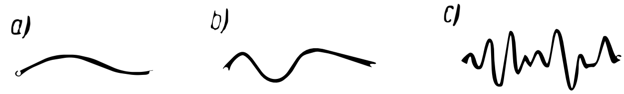


Fig. 3.

Let us assume a set-theory system $\mathcal{R}$ and define the following transformation

$$\Gamma(\mathcal{R}) = \sum_{r \in R} \Gamma(r), \tag{13}$$

which maps each branch $r = \langle x_r, y_r \rangle$ of the system in the non-empty subset S_r of one-dimensional simplexes determined in the Euclidean space E^n. with the vertices at the points x_r and y_r. The transformation Γ will be called the geometrical representation of the set-theory system $\mathcal{R}$

Denoting

$$S = \Gamma(\mathcal{R})$$

we define then the transformation Γ as

$$S = \Gamma(\mathcal{R}) \Rightarrow \Gamma(r) = S_r \subset S, \quad S_r \neq \theta, \tag{14}$$

where S_r is a set of one-dimensional simplexes with the following properties

$$s_r \in S_r \equiv s_r = (x_r, y_r), \quad r = \langle x_r, y_r \rangle. \tag{14a}$$

Now we introduce the following definition:

Definition 4. *The set S of one-dimensional simplexes determined by Eqs. 14 and* 14a *is called the topological system spread on the structural relation* $\mathcal{R}$

An example of a topological system is for instance a graph (multigraph) made up of the force lines of an electric field existing between two charges: positive and negative (Fig. 4a). Another example of a topological system is the network shown in Fig. 4b.

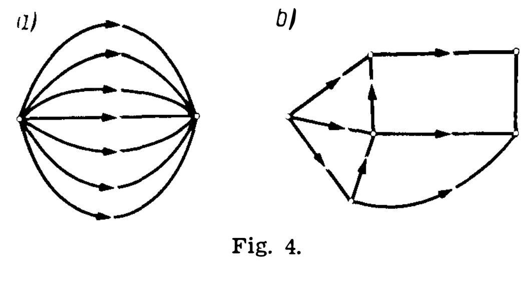


Fig. 4.

A system is spoken of as an oriented system if all the simplexes belonging to it are oriented simplexes. Otherwise we speak of a non--oriented system. In the case of a non-oriented system certain vertices are assumed to be conventional inputs or — conventional outputs.

2.3.2. Notion of the Structure of a System

Let us now discuss the definition of a topological structure of a system. Two systems S_1 and S_2 will be called *homeomorphic* if there exists a function f

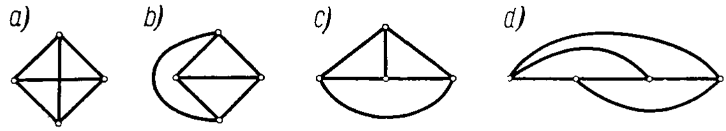


Fig. 5.

$$f(S) = \sum_{s \in S} f(s), \tag{15}$$

which is different-valued, continuous, whose inverse f^{-1} is also a continuous function, and which maps the system S_1 into S_2:

$$S_2 = f(S_1). \tag{16}$$

Let us now divide all topological systems into separate classes by placing in the same class those systems which are homeomorphic. *Such classes will be called topological structures and will be denoted by the symbol* $\bar{S}$. The following equality is then assumed:

$$\bar{S}_1 = \bar{S}_2 \equiv \text{(the system } S_1 \text{ and } S_2 \text{ are homeomorphic)} \tag{17}$$

Fig. 5. shows an example of four different systems having the same structure. Such a structure will be called *the bridge structure.*

Elementary structures are presented in Fig. 6. Those are the structures of *path, cycle, star* and *dendrite,* respectively.

2.3.3. Classes of Similarity of Topological Systems

The electrical engineer specializing in the theory of electric systems is well aware that a system with determined dynamical characteristics, for example an equilizer or electric filter, can be realized in different manners by means of systems with various structures, such as ladder structure, bridge structure, etc. This follows from the general principle according to which the systems having different topological structures can be characterized by analogical properties, for instance, they can transform input signals in an identical manner. Therefore, when dealing with problems concerned with the general synthesis of systems, we are less interested in obtaining a certain structure, than in defining a class of structures satisfying the conditions set forth in advance. It appears that for a broad domain of problems pertaining to the analysis and synthesis of systems, an appropriate definition of the above class of structures should take into account an investigation of how those structures are expanded into dendrites. The following discussion is devoted to this problem.

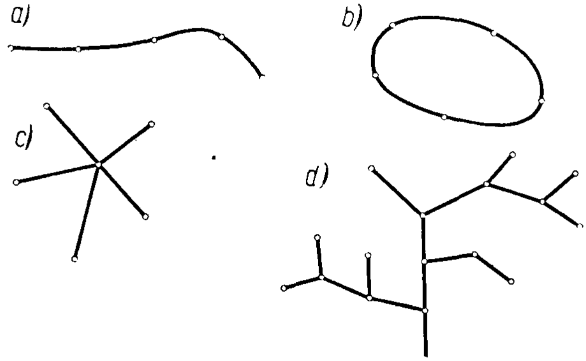


Fig. 6.

First of all it is necessary to introduce the notion of a *determined topological system.*

A topological system, the branches of which are set into one-to-one relation with natural numbers so that each branch corresponds to another number, will be called the determined system. The function setting into one-to-one relation the branches with the above natural numbers will be called the *describing function.* From the mathematical standpoint a determined system can be considered to be an ordered pair,

$$\mathcal{A} = \langle S, f \rangle, \tag{18}$$

consisting of the connected set of one-dimensional simplexes S and the describing function f.

A determined dendrite will be called a tree. Two trees

$$T_1 = \langle S_1, f_1 \rangle \text{ and } T_2 = \langle S_2, f_2 \rangle \tag{19}$$

will be spoken of as similar trees, and denoted $T_1 \sim T_2$ *if* $f_1(S_1) = f_2(S_2)$, *that is*

$$T_1 \sim T_2 \equiv f_1(S_1) = f_2(S_2). \tag{20}$$

Fig. 7 shows some examples of similar trees.

Let us now introduce an important definition.

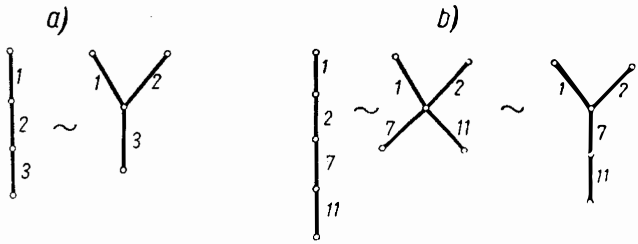


Fig. 7.

Definition 5. Two determined topological systems

$$\mathcal{A} = \langle S_1, f_1 \rangle \text{ and } \mathcal{B} = \langle S_2, f_2 \rangle \tag{21}$$

will be called similar and denoted by $\mathcal{A} \sim \mathcal{B}$, *if their expansions into trees contain exclusively similar trees.*

The class of all similar determined systems will be designated by the symbols $\overline{\mathcal{A}}, \overline{\mathcal{B}}, \ldots$ and will satisfy the following condition

$$\overline{\mathcal{A}} = \overline{\mathcal{B}} \equiv (\text{the system } \mathcal{A} \text{ and } \mathcal{B} \text{ are similar}). \tag{22}$$

Fig. 8 presents example of two systems with similar structures, and their respective expansions into trees.

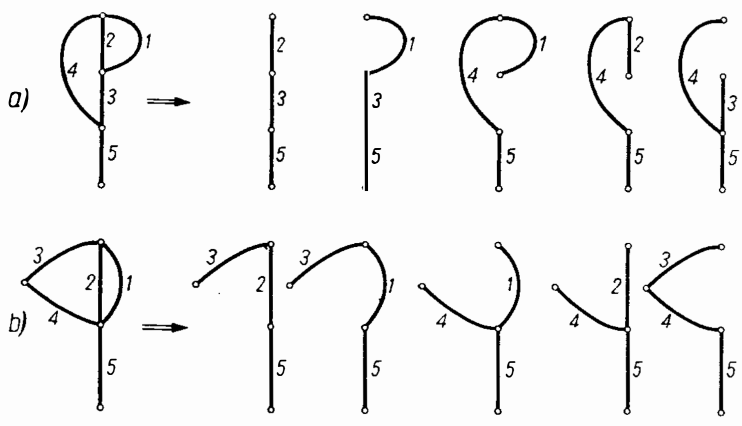


Fig. 8.

2.4. Concrete model of a system

In order to specify completely the concept of a system, we still have to define *a concrete model* which is the physical representation of a system. Since the main purpose of the present paper is to determine methods of topological analusis and synthesis of systems, we shall confine ourselves to presenting the definition of concrete system.

A concrete system is obtained from a topological system by setting into one-to-one relation certain physical quantities with its vertices (nodes) and branches.

Thus we determine the following functions

$$f_1: M \to X, \ f_2: S \to Y, \tag{23}$$

where X and Y are the sets of physical quantities, that is, of quantities having dimensions in a given system of units, for instance, in the c.g.s. system.

In the case of an electric system X is *the set of potentials,* and Y is *the set of currents.* The elements of the sets X and Y are, in a general case, functions of time and are called *the signals.*

The chapter concerning formalization of the concept of a system is concluded here, and we are going to pass to the algebra of structural numbers, which is a useful method in solving the problems of analysis and synthesis of linear systems.

3. METHOD OF STRUCTURAL NUMBERS

3.1. Foundations of the algebra of structural numbers

In order to solve the problems involved in the topological analysis and synthesis of systems, it is necessary to have a proper calculation algorithm. Such an algorithm can, of course, be constructed in different ways. The algorithm should, however, make it possible to solve the required problems in a possibly effective and simple way, this being the essential requirement. The algebra of structural numbers presented in this chapter creates a basis for constructing such an algorithm characterized by economy of work and calculations in comparison to other existing methods of calculation.

The notion of structural number is somewhat similar to that of matrix, which is widely applied in electrical engineering. This similarity is, however, merely apparent, since the calculus of structural numbers is based on quite different definitions of operations from those in the matrix calculus.

We shall call the structural number the system A of natural numbers arranged in the following table

$$A = \left[\begin{matrix} \alpha_{11}, \alpha_{12}, \dots \alpha_{1n} \\ \alpha_{21}, \alpha_{22}, \dots \alpha_{2n} \\ \vdots \quad \vdots \quad \quad \vdots \\ \alpha_{m1}, \alpha_{m2}, \dots \alpha_{mn} \end{matrix}\right]. \tag{24}$$

This system will be considered as a set of *columns*

$$C_k = \{\alpha_{1k}, \alpha_{2k}, \dots \alpha_{mk}\}; \qquad \alpha_{ik} \neq \alpha_{jk}, (i \neq j) \tag{25}$$

which in turn are unordered sets of natural numbers. The columns — as sets — will be considered as equal if they contain the same elements (independently of the succession of elements). It is assumed that in a structural number no identical columns can appear.

For structural numbers the operations are defined as follows:
Definition of the equality. *Two structural numbers are said to be equal if they contain the same columns.*

Example:

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}.$$

Definition of the sum. *The sum of two structural numbers A and B is the structural number containing all the columns of the number A and B, except for the identical columns (moreover, containing no other columns).*

Example

$$\begin{bmatrix} 2 & 3 & 7 \\ 7 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 7 & 2 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & \cancel{7} & 3 & \cancel{7} & 2 \\ 7 & 4 & \cancel{3} & 5 & \cancel{3} & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 7 & 4 & 5 & 4 \end{bmatrix}.$$

Definition of the product. *The product of two structural numbers A and B is the structural number, the columns of which are the sums (in the sense of the algebra of sets) of all the possible combinations of the columns of the numbers A and B, except for the maximal even number of identical columns, and for such columns where any element is repeated (moreover, the product contains no other columns).*

Example

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 5 & 1 & 3 \\ 6 & 2 & 5 \end{bmatrix} = \begin{bmatrix} \cancel{2} & 1 & \cancel{2} & \cancel{2} & \cancel{1} & \cancel{2} & \cancel{2} & 1 & \cancel{2} \\ \cancel{3} & 2 & \cancel{6} & \cancel{3} & \cancel{2} & \cancel{6} & \cancel{3} & 2 & \cancel{6} \\ \cancel{5} & 5 & \cancel{5} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{3} & 3 & \cancel{3} \\ \cancel{6} & 6 & \cancel{5} & \cancel{2} & \cancel{2} & \cancel{2} & \cancel{5} & 5 & \cancel{5} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 5 & 3 \\ 6 & 5 \end{bmatrix}.$$

Of course it is necessary to distinguish the structural number $[\theta]$ containing one and only one column, which is the empty set θ, from the structural number [ ] containing no column. It can be observed that the number [ ] is the modulus of addition, since for any structural number A it satisfies the equation

$$A + [\,] = A, \qquad A[\,] = [\,]. \tag{26}$$

Accordingly we shall simply denote the number [ ] by the symbol 0 and write

$$[\,] = 0. \tag{27}$$

The number $[\theta]$, on the other hand, is the modulus of multiplication, as it satisfies the equation

$$A[\theta] = A, \tag{28}$$

and consequently it will be denoted by the symbol 1

$$[\theta] = 1. \tag{29}$$

It can easily be noticed that the equation

$$AB = 0 \tag{30}$$

does not imply $A = 0$ or $B = 0$, and consequently the set of structural numbers has zero divisors. A pair of structural numbers $\langle A, B \rangle$ satisfying Equation (30) will be called a singular pair.

Example. A singular pair consists of the following structural numbers

$$A = \begin{bmatrix} 1\,3 \\ 2\,2 \end{bmatrix}, \quad B = \begin{bmatrix} 1\,2 \\ 3\,4 \end{bmatrix},$$

since we have

$$\begin{bmatrix} 1\,3 \\ 2\,2 \end{bmatrix} \begin{bmatrix} 1\,2 \\ 3\,4 \end{bmatrix} = 0.$$

Evidently the number $[\,] = 0$ constitutes a singular pair with any structural number.

Let us take into consideration two arbitrary structural numbers A and B. From the definition of the equality and the sum it follows that there is one and only one structural number X satisfying the equation

$$B + X = A.$$

This number will be spoken of as the difference of the structural numbers A and B.

$$X = A - B.$$

The operation determining the difference of structural numbers is called substraction. It can easily be noticed that the difference of the structural numbers A and B is the number $X = A + B$ and the following relation then holds:

$$A - B = A + B. \tag{31}$$

From Eq. (31) there follows the conclusion that in the set of structural numbers, substraction can always be substituted for addition. Substraction of structural numbers is then uniquely determined and always feasible; the set of structural numbers is therefore a set closed with respect to addition and substraction.

We are going to discuss now the following concepts connected with structural numbers: namely the concepts of

a) a geometrical image,
b) a complementary number and inverse image,
c) a determinant function,
d) an algebraic derivative,
e) a function of simultanity.

3.2. Some properties of the algebra of structural numbers

3.2.1. The geometrical image of a structural number

It is known from the theory of complex numbers that the image of a complex number is a point on the Gauss plane. But *the geometrical image of a structural number is the graph (topological system) all trees of which are determined by the columns of that number.* A geometrical image of the structural number is then *the class of similar determined systems.* For example the determined topological systems presented in Fig. 8 constitute the image of the following structural number A:

$$A = \begin{bmatrix} 2\,1\,1\,2\,3 \\ 3\,3\,4\,4\,4 \\ 5\,5\,5\,5\,5 \end{bmatrix}.$$

3.2.2. The complementary number and the inverse image of a structural number

If a set of elements occurring in a given structural number A is denoted by $\mathcal{A}$, then the complementary structural number will be the number A^d, the columns of which are differences (in the sense of the algebra of sets)

$$\mathcal{A} - C_1, \quad \mathcal{A} - C_2, \ldots \mathcal{A} - C_n, \tag{32}$$

of the set A and the columns C_i of the number A.

Example:

$$A = \begin{bmatrix} 4\;2\;3 \\ 2\;4\;6 \\ 3\;7\;5 \end{bmatrix}, \quad A^d = \begin{bmatrix} 5\;3\;2 \\ 6\;5\;4 \\ 7\;6\;7 \end{bmatrix}. \tag{33}$$

For this case we have

$$\mathcal{A} = \{2, 3, 4, 5, 6, 7\}.$$

The inverse geometrical image of a structural number is the graph (topological system) all co-trees of which are determined by the columns of that number.

It can be noted that the image of a given complementary number A^d is simultaneously the inverse geometrical image of the number A.

Table I shows the images and inverse images of certain structural numbers. It may be noted that the inverse image of a structural number is a graph with *a dual structure* (in Cauer sense) with respect to the graph representing its image (for planar graphs).

3.2.3. The determinant function of a structural number

The following function is determined on the set of structural numbers:

$$\det_{z_{\alpha_{ik}} \in Z} A = \det_{z_{\alpha_{ik}} \in Z} \begin{bmatrix} \alpha_{11} \dots \alpha_{1n} \\ \alpha_{21} \dots \alpha_{2n} \\ \vdots \qquad \vdots \\ \alpha_{m1} \dots \alpha_{mn} \end{bmatrix} = \sum_{k=1}^{n} \prod_{i=1}^{m} z_{\alpha_{ik}}, \tag{34}$$

whare Z is the subets or the given complex numbers $z_{\alpha_{ik}}$. For simplicity we shall write $\det_Z A$ instead of $\det_{z_{\alpha_{ik}} \in Z} A$.

Example:

$$\det_z \begin{bmatrix} 1\,2\,3 \\ 2\,4\,7 \\ 3\,5\,1 \end{bmatrix} = z_1 z_2 z_3 + z_2 z_4 z_5 + z_3 z_7 z_1 ;$$

$$z_1, z_2, \dots z_7 \in Z.$$

The establishment of the determinant function on the set of structural numbers entails important practical consequences. Owing to the determinant function we can, for instance, examine the properties of concrete systems whose topological structure is determined by a given structural number.

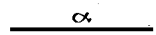
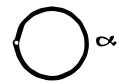
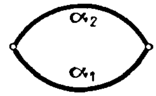
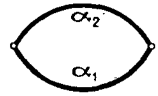
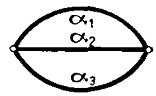
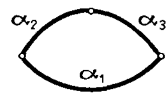
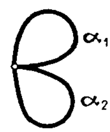
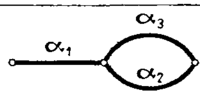
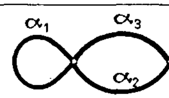
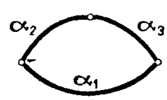
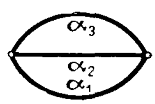
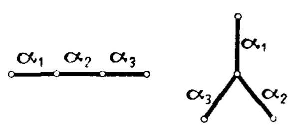
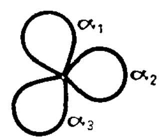
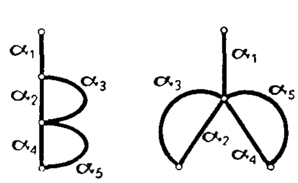
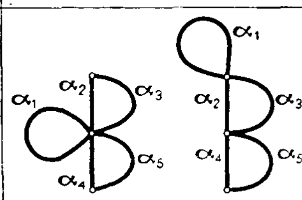
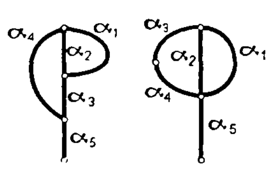
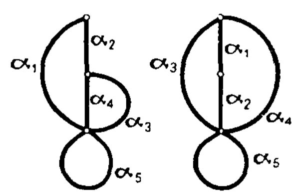
3.2.4. The algebraic derivative of a structural number

The algebraic derivative of a structural number A is determined by the following formula

$$\frac{\partial A}{\partial \alpha} = A \left| \begin{array}{l} \textit{columns not containing } \alpha \textit{ being omitted,} \\ \textit{and the element } \alpha \textit{ being omitted.} \end{array} \right. \tag{35}$$

It can be noted that the following relation holds

$$\det_Z \frac{\partial A}{\partial \alpha} = \frac{\partial}{\partial z_\alpha} [\det_Z A]; \qquad z_\alpha \in Z. \tag{36}$$

No.	Structural numbers	Geometric images	Geometric inverse images
1	$[\alpha]$		
2	$[\alpha_1, \alpha_2]$		
3	$[\alpha_1, \alpha_2, \alpha_3]$		
4	$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$		
5	$\begin{bmatrix} \alpha_1, \alpha_1 \\ \alpha_2, \alpha_3 \end{bmatrix}$		
6	$\begin{bmatrix} \alpha_1, \alpha_1, \alpha_2 \\ \alpha_2, \alpha_3, \alpha_3 \end{bmatrix}$		
7	$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$		
8	$\begin{bmatrix} \alpha_1, \alpha_1, \alpha_1, \alpha_1 \\ \alpha_2, \alpha_2, \alpha_3, \alpha_3 \\ \alpha_4, \alpha_5, \alpha_4, \alpha_5 \end{bmatrix}$		
9	$\begin{bmatrix} \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3 \\ \alpha_3, \alpha_4, \alpha_3, \alpha_4, \alpha_4 \\ \alpha_5, \alpha_5, \alpha_5, \alpha_5, \alpha_5 \end{bmatrix}$		

Example:

$$A = \begin{bmatrix} 1\,1\,2\,9 \\ 2\,3\,5\,1 \\ 4\,2\,7\,8 \end{bmatrix}; \quad \frac{\partial A}{\partial 1} = \begin{bmatrix} 2\,3\,9 \\ 4\,2\,8 \end{bmatrix}; \quad \frac{\partial A}{\partial 2} = \begin{bmatrix} 1\,1\,5 \\ 4\,3\,7 \end{bmatrix}.$$

We establish now the following

Property 2. *An inverse image of a structural number* $\frac{\partial A}{\partial \alpha}$ *is the inverse image of a number* A *in which the branch* α *is erased* (Fig. 9).

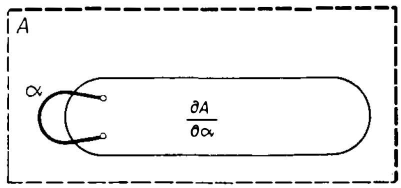


Fig. 9.

It is not difficult to prove that for any two structural numbers A_1 and A_2 the following relations hold

$$\frac{\partial}{\partial \alpha}(A_1 + A_2) = \frac{\partial A_1}{\partial \alpha} + \frac{\partial A_2}{\partial \alpha}, \quad \frac{\partial}{\partial \alpha}(A_1 \cdot A_2) = \frac{\partial A_1}{\partial \alpha} A_2 + \frac{\partial A_2}{\partial \alpha} A_1. \quad (37)$$

The notion of an inverse derivative

$$\frac{\delta A}{\delta \alpha}$$

is defined in the following manner:

$$\frac{\delta A}{\delta \alpha} = A \Big| \text{all the columns that contain the element } \alpha \text{ being omitted.} \quad (38)$$

Example

$$A = \begin{bmatrix} 1\,7\,2\,1 \\ 2\,5\,4\,5 \end{bmatrix}; \quad \frac{\delta A}{\delta 1} = \begin{bmatrix} 7\,2 \\ 5\,4 \end{bmatrix}; \quad \frac{\delta A}{\delta 5} = \begin{bmatrix} 1\,2 \\ 2\,4 \end{bmatrix}.$$

The inverse derivative has the following property:

Property 2. *An inverse image of a structural number* $\frac{\delta A}{\delta \alpha}$ *is the inverse image of a number* A *in which the branch* α *is short-circuited and erased.*

It can be noted that for any two structural numbers A_1 and A_2 the following relation holds:

$$\frac{\delta}{\delta \alpha}(A_1 + A_2) = \frac{\delta A_1}{\delta \alpha} + \frac{\delta A_2}{\delta \alpha}, \quad \frac{\delta}{\delta \alpha}(A_1 \cdot A_2) = \frac{\delta A_1}{\delta \alpha} A_2 + \frac{\delta A_2}{\delta \alpha} A_1 + A_1 \cdot A_2. \quad (39)$$

3.2.5. The simultaneous function

The term simultaneous function of a given structural number A (the inverse image of which contains oriented branches α and β) is given to the function denoted by the symbol

$$\operatorname*{Sim}_{Z}\left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta}\right); \qquad z_{\alpha_{ik}} \in Z \tag{40}$$

and determined as follows:

1. *The function* $\operatorname*{Sim}_{Z}\left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta}\right)$ *is a linear combination with the coefficients* $+1$ *and* -1 *of the terms occurring simultaneously in the functions* $\operatorname*{det}_{Z}\frac{\partial A}{\partial \alpha}$ and $\operatorname*{det}_{Z}\frac{\partial A}{\partial \beta}$.

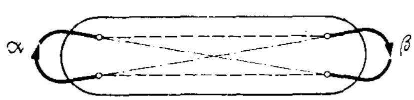


Fig. 10.

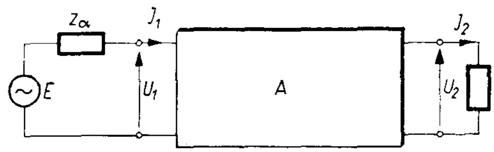


Fig. 11.

2. *If — after erasing from the inverse image of the number A the branches determined by the elements of a given term — we obtain a cycle with congruous (non-congruous) orientations of the branches* α *and* β, *then that term should be assigned the coefficient* $+1$ (-1) (Fig. 10).

The fundamental notions and properties of structural numbers established in the present chapter are a basis for the use of the algebra of structural numbers in the problems of analysis and synthesis of electric systems.

3.3. Analysis of electric systems by the method of structural numbers

By means of the notions of determinant function and simultaneous function, we can express any functions characterizing a system, for instance, input impedance, voltage transfer function, current transfer function, voltage-current transfer function, or composite transfer coefficient.

Let us consider a passive four-terminal network (Fig. 11).

For such a four-terminal network the following characteristic functions can be determined:

$$K_u = \frac{U_2}{E}, \quad K_i = \frac{I_2}{I_1}, \quad Z_1 = \frac{E}{I_1}, \quad \Gamma_{eff} = \ln \frac{E}{2U_2}\sqrt{\frac{z_\alpha}{z_\beta}}. \tag{41}$$

In terms of the theory of structural numbers we obtain the following formulae:

$$K_u = \frac{\underset{Z}{\operatorname{Sim}}\left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta}\right)}{\underset{Z}{\det} A} z_\beta ; \qquad K_1 = \frac{\underset{Z}{\operatorname{Sim}}\left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta}\right)}{\underset{Z}{\det} \frac{\partial A}{\partial \alpha}} ;$$

$$Z_1 = \frac{\underset{Z}{\det} A}{\underset{Z}{\det} \frac{\partial A}{\partial \alpha}} ; \qquad \Gamma_{eff} = \ln \left[\frac{\underset{Z}{\det} A}{2 \underset{Z}{\operatorname{Sim}}\left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta}\right)} \frac{1}{\sqrt{z_\alpha z_\beta}} \right]. \tag{42}$$

In all the above formulae A is a structural number, the inverse image of which is the given four-terminal network. Z is the set of impedances of the system.

We also have the following relations

$$K_i = \frac{1}{Z_\beta \frac{\partial}{\partial Z_\alpha}\left(\frac{1}{K_u}\right)}, \qquad Z_1 = \frac{1}{K_u \frac{\partial}{\partial Z_\alpha}\left(\frac{1}{K_u}\right)} .$$

Calculation of the structural number A is carried out on the basis of the following:

Theorem 1. *The structural number A is equal to the product of $P_1, P_2, \ldots P_n$ of one-row structural numbers corresponding to all the linearly independent cycles of its inverse image.*

Example 1. The structural number A, the inverse image of which is represented in Fig. 12, is equal to the following product:

$$A = P_1 P_2 P_3 = [1\ 3\ 5][2\ 4\ 5][3\ 4\ 6].$$

Hence,

$$\begin{array}{rccc} & 1 & 3 & 5 \\ \times & 2 & 4 & 5 \\ & 3 & 4 & 6 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 4 & 4 & 5 & 5 & 5 & 2 & 2 & 4 & 5 & 5 & 2 & 2 & 2 & 4 & 4 \\ 3 & 4 & 6 & 3 & 6 & 3 & 4 & 6 & 4 & 6 & 6 & 4 & 6 & 3 & 4 & 6 & 3 & 6 \end{bmatrix}$$

Also $A = [1\ \ 3\ \ 5]\ [1\ \ 2\ \ 6]\ [1\ \ 2\ \ 3\ \ 4]$; etc.

A structural number can also be determined simply from knowledge of the graph which is its image. We make use, then, of the following:

Theorem 2. *A structural number A with a given geometrical image, having n vertices is equal to the product $P_1, P_2, \ldots P_{n-1}$ of one-row structural numbers corresponding to $(n - 1)$ arbitrary vertices (nodes) of its image.*

Example 2. The structural number, the image of which is presented in Fig. 13, is

$$A = [1\ 2\ 3]\ [2\ 4\ 5]\ [5\ 6\ 7]\ [1\ 7].$$

Also

$$A = [3\ 4\ 6]\ [2\ 4\ 5]\ [5\ 6\ 7]\ [1\ 7]; \qquad \text{etc.}$$

Let us now take an example of analysis of an electric four-terminal network in which we use the algebra of structural numbers.

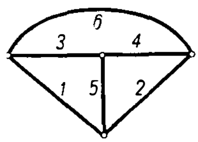


Fig. 12.

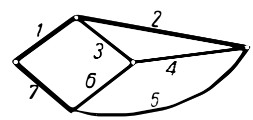


Fig. 13.

Example 3. Let us calculate the voltage and current transfer functions of the four-terminal network shown in Fig. 14.

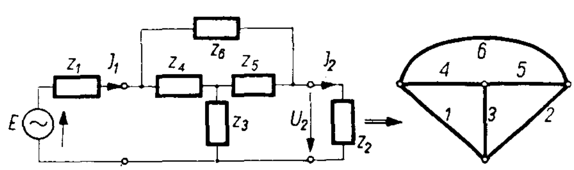


Fig. 14.

We have

$$A = [1\ 3\ 4]\ [2\ 3\ 5]\ [4\ 5\ 6]$$

We multiply according to the following scheme

$$\begin{array}{cccc} & 1 & 3 & 4 \\ \times & 2 & 3 & 5 \\ & 4 & 5 & 6 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & \cancel{3} & 3 & 4 & 4 & \cancel{4} & 4 & 4 \\ 2 & 2 & 2 & 3 & 3 & 3 & 5 & 5 & 2 & 2 & 2 & \cancel{5} & 5 & 2 & 2 & \cancel{3} & 3 & 5 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 6 & 4 & 5 & 6 & \cancel{4} & 6 & 5 & 6 & \cancel{5} & 6 & 6 \end{bmatrix}.$$

Hence

$$\frac{\partial A}{\partial 1} = \begin{bmatrix} 2 & 2 & 2 & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{5} & 5 \\ 4 & 5 & 6 & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{4} & 6 \end{bmatrix}.$$

$$\frac{\partial A}{\partial 2} = \begin{bmatrix} 1 & 1 & 1 & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{4} & 4 \\ 4 & 5 & 6 & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{5} & 6 \end{bmatrix}.$$

The columns occurring simultaneously in the above derivatives are distinguished by rectangles. The simultaneous function is then

$$\operatorname*{Sim}_{Z}\left(\frac{\partial A}{\partial 1'}\frac{\partial A}{\partial 2}\right) = Z_3 Z_4 + Z_3 Z_5 + Z_3 Z_6 + Z_4 Z_5 .$$

Therefore

$$K_u = \frac{(Z_3Z_4+Z_3Z_5+Z_3Z_6+Z_4Z_5)\,Z_2}{Z_1Z_2Z_4+Z_1Z_2Z_5+Z_1Z_2Z_6+Z_1Z_3Z_4+Z_1Z_3Z_5+Z_1Z_3Z_6+\ \dots\ +Z_4Z_5Z_6},$$

$$K_i = \frac{Z_3 Z_4 + Z_3 Z_5 + Z_3 Z_6 + Z_4 Z_5}{Z_2 Z_4 + Z_2 Z_5 + Z_2 Z_6 + Z_3 Z_4 + Z_3 Z_5 + Z_3 Z_6 + Z_5 Z_4 + Z_5 Z_6}$$

It can be seen from the above example that great advantages of calculation are gained by the use of the algebra of structural numbers in problems concerning analysis of systems.

3.4. Synthesis of electric systems by the method of structural numbers

The classical methods of synthesis of linear systems are prescriptive methods. They require an individual approach for every concrete problem of synthesis. The method of structural numbers makes it possible to solve the problem of synthesis in a very general manner without imposing any restrictions on the structure of the system designed. Such a problem — due to a high degree of complication in calculations — should be solved by a digital computer.

The problem of synthesis of an electric network can be split into two stages, namely,

1) topological synthesis of a graph,
2) calculation of the value of the individual elements in the system.

The term topological synthesis of a graph is understood as a procedure aiming at determinig the class of structures of similar graphs which make up the geometric image (or the inverse image) of a given structural number A. From Theorem 1 and 2 cited above it follows immediately that such a procedure would lead to determining all the possible expansions of a structural number A into one-row prime factors. A structural number can, however, have no geometric representation of itself. In this connection, the following conditions are specified as necessary and sufficient for a structural number to have a geometric image.

1. *The number A must have the expansion into one-row prime factors*

$$A = P_1 P_2 \dots P_m . \tag{43}$$

2. *An arbitrary element* α_{ik} *of the numbers* $P_1, \ldots P_m$ *may occur at most in two numbers* P_i, P_j *of the product* (43).

Moreover, the following additional conditions must be satisfied

$$3.\ P_i \neq P_j; \quad i, j = 1, 2, \ldots m; \quad (i \neq j),$$
$$4.\ P_i \neq \sum_k P_k; \quad k = 1, 2, \ldots m; \quad (k \neq i). \tag{44}$$

Otherwise we would have obtained $A = 0$.

The activities connected with the synthesis of the two-terminal network R, L, C, which would be programmed for a computer, are set up below as an example.

Example. Let us determine a family of structures of systems realizing the impedance of the R, L, C two-terminal network.

In terms of the method of structural numbers, the impedance of the two-terminal network can be determined by means of the following formulae [2]:

$$Z = \frac{\underset{Y}{\det} \dfrac{\partial A}{\partial \alpha}}{\underset{Y}{\det} A}; \quad Z = z_\alpha \frac{\underset{Z}{\det} \dfrac{\delta A^d}{\delta \alpha}}{\underset{Z}{\det} A^d}, \tag{45}$$

where A is a structural number, the image of which is the graph of the system, z_α — is the impedance of the branch which is between the terminals of our network (Fig. 15).

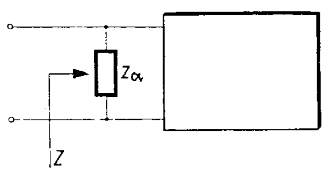


Fig. 15.

In our further considerations we are going to make use of the second formula which expresses the impedance Z in terms of the impedances z_i of the branches of our network.

Let us assume that in each branch of the network there will occur a series connection of the resistor, of the induction coil, and of the condenser. Thus

$$Z_i = sL_i + R_i + s^{-1} C_i^{-1}. \tag{46}$$

---

[2] We assume that the two-terminal network does not contain shrot-circuited elements.

Assume that the impedance, the synthesis of which is being carried out, is determined by means of the following real, positive function

$$Z(s) = \frac{a_{n+1}\, s^{n+1} + \ldots + a_0 + \ldots + a_{-(n+1)}\, s^{-(n+1)}}{b_n\, s^n + \ldots + b_0 + \ldots + b_{-n}\, s^{-n}} = \frac{P(s)}{Q(s)}. \tag{47}$$

We introduce the following denotations:

b — number of branches in the network,

w — number of nodes in the network.

It can be seen that in the case of the realization of the system on the basis of the second formula (45), the degree of the denominator $Q(s)$ is equal to $(b - w + 1)$; hence

$$b - w + 1 = n. \tag{48}$$

Since, in the process of determining $3b$ unknown values of the elements of the system we obtain $(4n + 4)$ algebraic equations, the following inequality must be satisfied:

$$3b \geqslant 4n + 4, \tag{49}$$

which together with the formula (48) yields the following evaluation for the number of nodes in the network

Since the number m of the factors in the product (49) is equal to $w - 1$, we can generally assume that

$$m = E\left(\frac{n+4}{3}\right) + k: \quad k = 1, 2, \ldots \tag{51}$$

where $E(x)$ denotes the *entire* x.

In connection with the above, the network will have the number of nodes as follows

$$b = E\left(\frac{n+4}{3}\right) + k + n; \quad k = 1, 2, \ldots \tag{52}$$

The activities connected with carrying out the synthesis are specified as follows:

1. We assume that $m = E\left(\frac{n+4}{3}\right) + 1$ and consider the following set of branches

$$\mathfrak{B} = \{1, 2, \ldots b\}; \quad b = E\left(\frac{n+4}{3}\right) + n + 1$$

in terms of which we set up the products of one-row structural numbers

$$A = P_1 P_2 \ldots P_m,$$

according to the principles of realizability given formerly (43) and (44), with the assumption that the numbers P_l consist of at least two elements.

For example:

$$A = [1\ 2][2\ 3][3\ 4\ 5] \ldots \text{etc.}$$

By carrying out multiplication we obtain all the possible structural numbers A_1, A_2 $A_3, \ldots A_i, \ldots$ and thus all the possible graphs corresponding to $k = 1$.

2. We find all the complementary structural numbers

$$A_1^d, A_2^d, \ldots A_i^d, \ldots$$

3. We calculate the inverse algebraic derivatives

$$\frac{\delta A_i^d}{\delta 1}, \frac{\delta A_i^d}{\delta 2}, \ldots \frac{\delta A_i^d}{\delta \alpha}, \ldots \frac{\delta A_i^d}{\delta b}.$$

4. We find the determinant functions

$$\det_z A_i^d, \det_z \frac{\delta A_i^d}{\delta \alpha}; \qquad \alpha = 1, 2, \ldots b.$$

5. By comparing the coefficients of the rational functions

$$\frac{P(s)}{Q(s)} = \frac{\det_z \frac{\delta A_i^d}{\delta \alpha}}{\det_z A_i^d} z_\alpha, \tag{53}$$

we obtain the following sets of nonlinear equations [3)]

$$\begin{aligned} &f_{n+1}^{\alpha, i}(R_1, \ldots L_1, \ldots C_1, \ldots) = a_{n+1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_{-(n+1)}^{\alpha, i}(R_1, \ldots L_1, \ldots C_1, \ldots) = a_{-(n+1)}, \\ &\varphi_n^i(R_1, \ldots L_1, \ldots C_1, \ldots) = b_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\varphi_{-n}^i(R_1, \ldots L_1, \ldots C_1, \ldots) = b_{-n}. \end{aligned} \tag{54}$$

These sets of equations can be solved by a digital computer for example, by means of the relaxation or iterative method; we obtain then the values of the elements in the particular branches of the network, the structures of which have been determined by the structural numbers $A_1, A_2, \ldots A_i, \ldots$

It can be seen that the number of unknowns is greater than the number of equations by the quantity δ:

$$\delta = 3E\left(\frac{n+4}{3}\right) - n - 1.$$

In this connection the values of δ elements in the system (e.g. inductances) are assumed to be zero.

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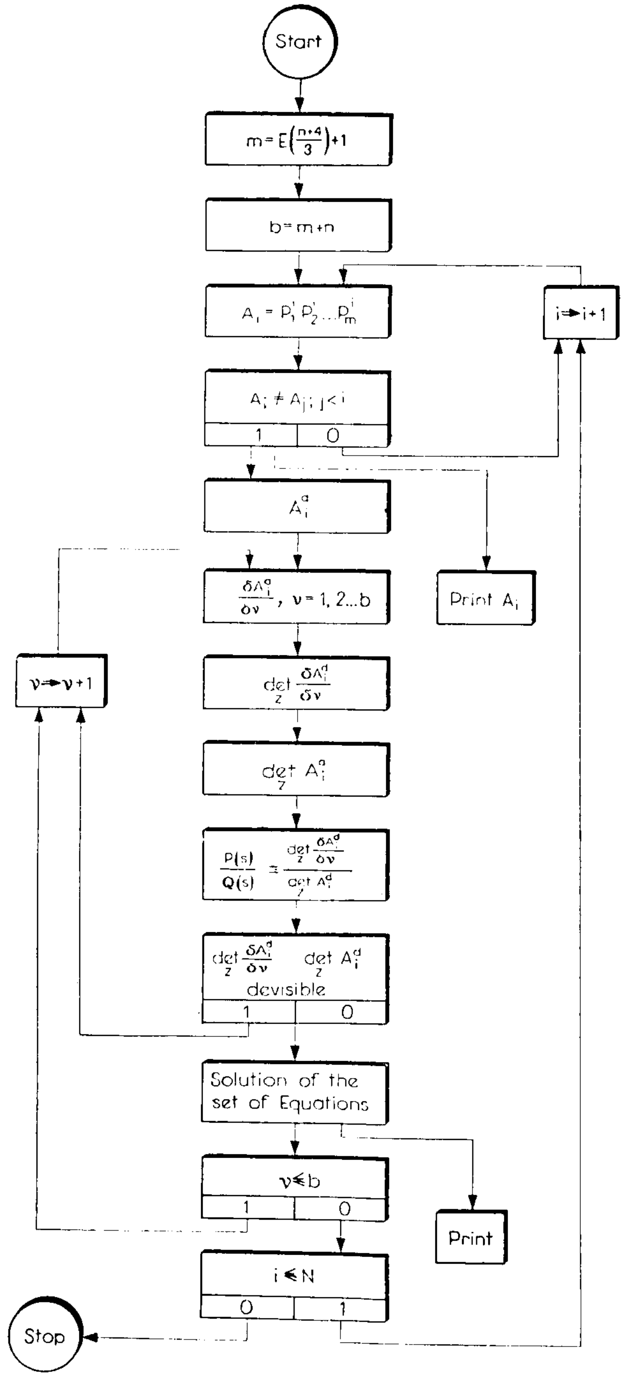
3) We disregard such structural numbers as A_i^d, for which the nominator and denominator of the expression (53) are divisible.


Start
m = E((n+4)/3)+1
b = m+n
A_i = P_1^i P_2^i ... P_m^i
i → i+1
A_i ≠ A_j ; j < i
1
0
A_i^d
Print A_i
δA_i^d/δν , ν = 1, 2...b
ν → ν+1
det_z δA_i^d/δν
det_z A_i^d
P(s)/Q(s) = det_z (δA_i^d/δν) / det_z A_i^d
det_z δA_i^d/δν det_z A_i^d
devisible
1
0
Solution of the set of Equations
ν ≤ b
1
0
Print
i ≤ N
0
1
Stop

We can also assume additional equations determining the relations between the elements of the system. For instance, we may assume that all the inductances occurring in the two-terminal network being designed are equal to one another.

6. If necessary, we repeat the entire cycle of calculations for

$$m = E\left(\frac{n+4}{3}\right) + 2 \qquad m = E\left(\frac{n+4}{3}\right) + 3, \text{etc.}$$

It is clear that the system now obtained will contain a greater number of branches than did the systems calculated formerly for $k = 1$.

Table II shows a simplified block diagram for a computer, which includes all the operations connected with carrying out the synthesis of th *R, L, C* two-terminal network with a minimum number of branches.

The results of the calculations can be divided into two separate groups. All these systems in which we have obtained positive *R, L, C* elements may be presented in the first group. The remaining systems, realizable with the use of active elements, will then belong to the second group.

Appendix

1. Proof for formulae (42)

Let us prove the correctness of the formula determining the current transfer function K_i. This transfer function — for the four-terminal network presented in Fig. 10 — can be found in terms of the following formula

$$K_i = \frac{I_2}{I_1} = \frac{\sum\limits_{\nu} I_{2\nu} - \sum\limits_{\mu} I_{2\mu}}{I_1},$$

where

$I_{2\nu}$ are currents flowing in the four-terminal network through the individual circuits containing the branches $\alpha = 1$ and $\beta = 2$ which have the same orientation,

$I_{2\mu}$ are currents flowing in the four-terminal network through the individual circuits containing the branches α and β which have not the same orientation.

On the other hand, since the columns of the number A determine all the co-trees in the system, then the columns of the numbers $\frac{\partial A}{\partial \alpha}$ and $\frac{\partial A}{\partial \beta}$ determine those branches in the system, the removal of which converts the system into a network with one cycle. Identical columns of the numbers $\frac{\partial A}{\partial \alpha}$ and $\frac{\partial A}{\partial \beta}$ determine then all those branches the removal of which leads to cycles containing the branches α and β, that is, leads to the circuits of the currents $I_{2\nu}$ and $I_{2\mu}$.

Moreover, making use of Maxwell equations we get

$$\sum_{i=1}^{n} z_{ij} I_i = \begin{cases} E; & j = 1 \\ 0; & j > 1 \end{cases} \tag{1}$$

$$j = 1, 2, \ldots n.$$

We can easily find that

$$K_i = \frac{I_2}{I_1} = \frac{\Delta_{12}}{\Delta_{11}},$$

where Δ_{11} and Δ_{12} are the corresponding subdeterminants of the system (1).

Clearly, Δ_{11} is the sum of all the values of the co-trees, if the system with the branch $1 = \alpha$ being erased. Thus

$$\Delta_{11} = \det_{Z} \frac{\partial A}{\partial \alpha},$$

where A is a structural number the inverse image of which is the given system.

Δ_{12} is a linear combination with the coefficients $+1$ and -1 of such co-trees of the system with the erased branch α or β, the removal of which implies $K \neq 0$. After removing any of the above co-trees, the system is reduced to a a cycle constituting the circuit of the current $I_{2\nu}$ or $I_{2\beta}$.

Summarizing our reasoning we state that

$$\Delta_{12} = \operatorname*{Sim}_{Z} \left(\frac{\partial A}{\partial \alpha}, \frac{\partial A}{\partial \beta} \right),$$

where Z is the set of impedances of the system.

Similarly, we can prove the correctness of the remaining formulae (42).

2. Proof for Theorem 2

It can immediately be seen that the theorem is satisfied for a system with the structure of a cycle. It is also satisfied in the case of a system with two cycles. In fact, such a system can always be simplified to the form shown in Fig. 1-A a or b, where 1, 2 and 3 are the sums of the corresponding elements of the system. In the case of the system a) we have

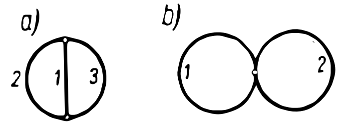
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{bmatrix}$$

and then in fact $A = [1\ 2]\,[1\ 3]$.

Also in the case b)

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [1]\,[2].$$

It can easily be proved that the above results are correct in the case where the branches 1, 2 and 3 consist of any number of elements.



Let us now assume that Theorem 1 is correct for a system with n independent cycles. By reasoning similarly as above we can easily prove that the theorem is correct for a system with $(n + 1)$ cycles. Thus, by the principle of induction Theorem 1 is valid for a system with any number of cycles.

Theorem 2 can be proved in a similar manner.

3. Proof for formulae (45)

Let us present a proof for the second formula of (45). There is a well-known theorem that the impedance of a networek measured between the nodes a, b i equal to the ratio

$$Z = \frac{\Delta_{ab}}{\Delta},$$

where Δ is the main determinant of the impedance matrix of the given network,

Δ_{ab} is the main determinant of the impedance matrix of the network with short-circuited nodes a and b.

Moreover, we have (for the system presented in Fig. 13).

$$\Delta = \det_{Z} A\,; \qquad \Delta_{ab} = z_{\alpha} \det_{Z} \frac{\delta A}{\delta \alpha},$$

where the formula for Δ_{ab} results from the property 2.

In similar manner, we may present a proof for the first formula of (45).

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Translated by I. Bellert

SYNTHESIS OF AN ARTIFICIAL LONG LINE AND AN AMPLIFIER WITH A NEGATIVE IMPEDANCE BY MEANS OF PADE'S APPROXIMATION METHOD [1])

N 65 - 36013

ABSTRACT

Two problems are discussed in the paper; the first concerns the synthesis and design of a passive electric system realizing frequency patterns of a transmission long line — that is, the characteristics of the impedance and the composite transfer coefficient of the long line. A system designed on the basis of the method presented realizes long line characteristics in a frequency band six times wider than does a similar system designed by the method of Zobel [3], the same number of elements being used.

The second problem concerns a method for designing a nonreflecting four-terminal amplifier with a negative impedance (negistor), which compensates the attenuation and amplitude-phase distortions of a long line. This method, which is based on the conclusions from the first part of the paper, makes it possible, for example, to design a system decreasing the attenuation of a transmission long line 15 km in length within a band up to 6 kHz.

Author

1. INTRODUCTION

By the term artificial long line, is meant an electrical fourpole which realizes the frequency patterns of composite transfer coefficients and wave impedances of a transmission long line within a given frequency band. Artificial lines are indispensable in every up-to-date wire transmission laboratory. There are not in the professional literature many papers devoted to the problem of desingning artificial lines. A paper of essential importance is that by Zobel [3] in which an effective method is presented for realizing an artificial line in terms of a passive four-pole of the X type. The main drawback of Zobel's method is the manner assumed of approximating the composite transfer coefficient of the long line by means of which is obtained a narrow band of frequencies reproduced by the long line. This method is then inefficient from a practical point of view.

In the present paper the manner of realizing a long line is a synthesis method. The approximation of the exponential function — which represents the transfer function of the long line by a rational function — is effected on the basis of an approximation according to Pade [2]. Owing

[1]) Archiwum Elektrotechniki (Vol. VIII, No. 4, 1959, pp. 595—615).

to this approximation, we obtain an almost six-fold broadening of the band of frequencies reproduced by the system as compared with the system designed by means of Zobel' method, the same number of elements being used.

We call the negator an amplifying system, symetric as regards energy transmission, in which the aplifying effect is obtained by means of negative impedances. The task of a negator is to compensate attenuation and to correct attenuation distorions brought about by a transmission line. Negistors are designed in a two-pole or four-pole system. The paper is concerned with a four-pole negistor designed on the basis of the method for an artificial line. Such type of a negistor corrects attenuation distortions introduced by the line as well as phase distortions, and moreover does not bring about any reflections of energy.

2. SYNTHESIS OF AN ARTIFICIAL LONG LINE

Let us take as basis of our considerations the lattice four-pole shown in Fig. 1. The wave parameters of this four-pole are determined by the formulae

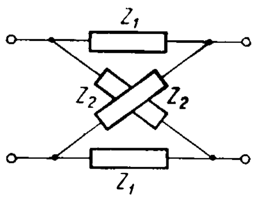

Fig. 1. Lattice four--pole

$$\tanh \frac{\Gamma}{2} = \sqrt{\frac{Z_2}{Z_1}}, \tag{1}$$

$$Z = \sqrt{Z_1 Z_2}, \tag{2}$$

where Γ is the wave composite transfer coefficient, and Z is the wave impedance of the four-pole.

From Eq. (1) we shall, by means of elementary transformations, obtain the formula determining the transfer function of the four-pole, corresponding with the matching conditions — that is, the quantity $e^{-\Gamma}$

$$e^{-\Gamma} = \frac{1 - \sqrt{\frac{Z_1}{Z_2}}}{1 + \sqrt{\frac{Z_1}{Z_2}}}. \tag{3}$$

An artificial long line will be called the four-pole, satisfying in a given frequency band the following equalities

$$Z = Z_T, \tag{4}$$

$$\Gamma = \Gamma_T, \tag{5}$$

where Z_T and Γ_T are the wave parametres of the transmission line.

In order to enable the condition (5) to be satisfied in a given frequency band, we shall approximate the transfer function $e^{-\Gamma}$ of the long line by means of the rational function

$$e^{-\Gamma_T} = e^{-\gamma_T l} = \frac{Q}{P} = \frac{a_0 - a_1 x + a_2 x^2 - \ldots + (-1)^n a_n x^n}{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}, \tag{6}$$

where $x = \gamma_T l = \Gamma_T$

The manner of approximating the exponential function $e^{-\Gamma_T}$ by means of the above rational function will be discussed later in the paper.

Comparing the function (3) with the function (6) — that is to say satisfying the condition (5) — we shall obtain

$$\frac{1 - \sqrt{\dfrac{Z_1}{Z_2}}}{1 + \sqrt{\dfrac{Z_1}{Z_2}}} = \frac{Q}{P}. \tag{7}$$

Hence

$$\sqrt{\frac{Z_1}{Z_2}} = \frac{P - Q}{P + Q}. \tag{8}$$

Multiplying and dividing Formulae (2) and (8) by their respective sides, we shall arrive at relations determining the impedances Z_1 and Z_2, namely

$$Z_1 = \frac{P - Q}{P + Q} Z, \tag{9}$$

$$Z_2 = \frac{P + Q}{P + Q} Z. \tag{10}$$

Therefore, assuming $Z = Z_T$, — that is, satisfying the condition (4), we find

$$Z_1 = \frac{a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{n-1} x^{n-1}}{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_n x^n} Z_T. \tag{11}$$

$$Z_2 = \frac{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_n x^n}{a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{n-1} x^{n-1}} Z_T. \tag{12}$$

The composite transfer coefficient and the wave impedance of a homogeneous long line are — as is well known — defined by the following formulae

$$\gamma_T = \sqrt{(R + SL)(G + SC)}; \qquad Z_T = \sqrt{\frac{R + SL}{G + SC}} \tag{13}$$

where $S = j\omega$

Taking then

$$Z_a = (R + SL)\,l\,; \qquad Z_b = \frac{1}{l\,(G + SC)}\,, \tag{14}$$

or

$$X = \gamma_T\,l = \sqrt{\frac{Z_a}{Z_b}}\,; \qquad Z_T = \sqrt{Z_a Z_b}\,,$$

we shall be able to write the functions Z_1 and Z_2 as follows

$$Z_1 = \frac{a_1\sqrt{\frac{Z_a}{Z_b}} + a_3\left(\sqrt{\frac{Z_a}{Z_b}}\right)^3 + \ldots + a_{n-1}\left(\sqrt{\frac{Z_a}{Z_b}}\right)^{n-1}}{a_0 + a_2\frac{Z_a}{Z_b} + \ldots + a_n\left(\frac{Z_a}{Z_b}\right)^{\frac{n}{2}}}\sqrt{Z_a Z_b}\,. \tag{15a}$$

$$Z_2 = \frac{a_0 + a_2\frac{Z_a}{Z_b} + \ldots + a_n\left(\frac{Z_a}{Z_b}\right)^{\frac{n}{2}}}{a_1\sqrt{\frac{Z_a}{Z_b}} + a_3\left(\sqrt{\frac{Z_a}{Z_b}}\right)^3 + \ldots + a_{n-1}\left(\sqrt{\frac{Z_a}{Z_b}}\right)^{n-1}}\sqrt{Z_a Z_b}\,; \tag{15b}$$

Consider now two particular cases — namely, the case where the degrees of the polynomials P and Q are equal to $n = 2$, and $n = 4$, respectively.

For $n = 2$ we shall have $a_3 = a_4 = \ldots = 0$, and

$$Z_1 = \frac{a_1 Z_a Z_b}{a_0 Z_b + a_2 Z_a}\,; \qquad Z_2 = \frac{a_0}{a_1} Z_b + \frac{a_2}{a_1} Z_a\,.$$

Hence

$$Z_1 = \frac{1}{\frac{1}{\frac{a_1}{a_0} Z_a} + \frac{1}{\frac{a_1}{a_2} Z_b}} = \frac{1}{\frac{1}{Z_1^{(1)}} + \frac{1}{Z_1^{(2)}}}\,, \tag{16a}$$

where

$$Z_2 = Z_2^{(1)} + Z_2^{(2)}\,, \tag{16b}$$

$$Z_1^{(1)} = \frac{a_1}{a_0} Z_a\,; \qquad Z_1^{(2)} = \frac{a_1}{a_2} Z_b\,,$$

$$Z_2^{(1)} = \frac{a_2}{a_1} Z_a\,; \qquad Z_2^{(2)} = \frac{a_0}{a_1} Z_b\,. \tag{17}$$

It is clear then that for this case the impedance Z_1 can be realized as a parallel, and the impedance Z_2 as a series connection of two elements (Fig. 2).

Consider now the case in which $n = 4$. In such a case $a_5 = a_6 = \ldots = 0$, and after elementary transformations

$$Z_1 = \frac{a_1 Z_a Z_b^2 + a_3 Z_a^2 Z_b}{a_0 Z_b^2 + a_2 Z_a Z_b + a_4 Z_a^2},$$
$$Z_2 = \frac{a_0 Z_b^2 + a_2 Z_a Z_b + a_4 Z_a^2}{a_1 Z_b + a_3 Z_a}. \tag{18}$$

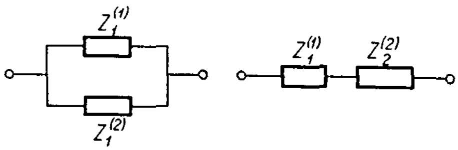

Fig. 2. Impedances Z_1 and Z_2 for the case in which the polynomials P and Q are of the second order

The expression determining Z_1 can be transformed as follows

$$Z_1 = \frac{(a_1 Z_b + a_3 Z_a) Z_a Z_b}{\frac{a_4}{a_3} Z_a (a_1 Z_b + a_3 Za) + \frac{a_0}{a_1} Z_b (a_1 Z_b + a_3 Z_a) + \left(a_2 - \frac{a_3 a_0}{a_1} - \frac{a_1 a_4}{a_3}\right) Z_a Z_b} =$$
$$= \frac{1}{\frac{1}{\frac{a_3}{a_4} Z_b} + \frac{1}{\frac{a_1}{a_0} Z_a} + \frac{1}{\frac{a_1 a_3}{a_1 a_2 a_3 - a_3^2 a_0 - a_1 2 a_4}(a_1 Z_b + a_3 Z_a)}} = \tag{19}$$
$$= \frac{1}{\frac{1}{Z_1^{(1)}} + \frac{1}{Z_1^{(2)}} + \frac{1}{Z_1^{(3)} + Z_1^{(4)}}}$$

In a similar manner, we can obtain for Z_2 the following expression

$$Z_2 = \frac{a_4}{a_3} Z_a + \frac{a_0}{a_1} Z_b +$$
$$+ \frac{1}{\frac{1}{\frac{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0}{a_1^2 a_3} Z_a} + \frac{1}{\frac{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0}{a_1 a_3^2} Z_b}} = \tag{20}$$
$$= Z_2^{(1)} = Z_2^{(2)} + \frac{1}{\frac{1}{Z_2^{(3)}} + \frac{1}{Z_2^{(4)}}}.$$

From Eqs. (19) and (20), there follow immediately the structures of two-poles yielding the impedances Z_1 and Z_2 (Fig. 3). and the following relations defining the values of the particular elements

$$\left.\begin{aligned}
&Z_1^{(1)} = \frac{a_3}{a_4} Z_b\,; \quad Z_1^{(2)} = \frac{a_1}{a_0} Z_a\,; \\
&Z_1^{(3)} = \frac{a_1^2 a_3}{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0} Z_b\,; \quad Z_1^{(4)} = \frac{a_1 a_3^2}{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0} Z_a\,; \\
&Z_2^{(1)} = \frac{a_4}{a_3} Z_a\,; \quad Z_2^{(2)} = \frac{a_0}{a_1} Z_b\,; \\
&Z_2^{(3)} = \frac{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0}{a_1^2 a_3} Z_a\,; \quad Z_2^{(4)} = \frac{a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0}{a_1 a_3^2} Z_b
\end{aligned}\right\} \qquad (21)$$

As a condition of realizing the elements $Z_1^{(3)}$, $Z_1^{(4)}$, $Z_2^{(3)}$, $Z_2^{(4)}$, it is necessary that the following inequalities be satisfied

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 \geqslant 0\,. \qquad (22)$$

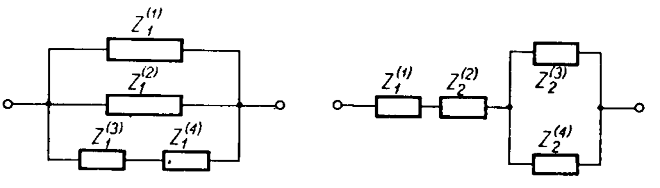

Fig 3. Impedances Z_1 and Z_2 for the case in which the polynomials P and Q are of the fourth order

We have thus the synthesis of a lattice four-pole whose transfer function is a rational function of the form (6), and whose wave impedances are

$$Z = \sqrt{Z_a Z_b}\,.$$

In fact, in order to decrease the number of elements in the four-pole, it is possible to make use of one of the structures equivalent to a lattice four-pole (Fig. 4). The structure (*C*) shown in Fig. 4 is more convenient in practice than the equivalent structure (*b*), since the former does not require the application of symmetrical transformers which cut off the path for d.c. current in the long line and bring about considerable distortions in the range of low frequencies. In the system (*C*), the coils located in horizontal branches are coupled magnetically with a view to preserving symmetry. It should be noted that the coils, not being ideally coupled magnetically, also introduce unavoidable distortions of the frequency pattern. Accordingly, if we want to realize with great accuracy the frequency patterns of a transmission long line, we should apply the basic system of the type X, without regard to the number of elements used.

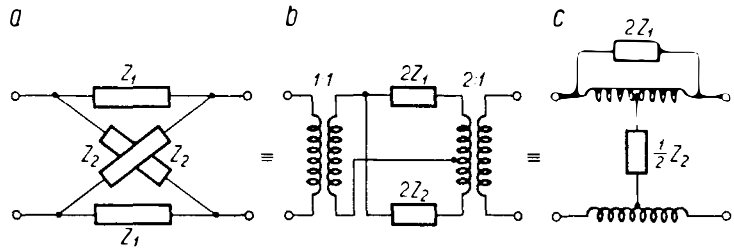

Fig. 4. Systems equivalent to a lattice four-pole

3. THE PROBLEMS OF APPROXIMATION

The manner of approximating the exponential function $e^{-\Gamma_T}$, determining the composite transfer coefficient of a transmission line by means of a rational function has a decisive influence on the accuracy of realizing the characteristic of the designed system obtained as a result of the synthesis.

It is therefore necessary to lay special emphasis on the question of approximation. Before discussing the manner of approximation, let us consider what should be the interval $0 \leqslant X \leqslant b$ of the approximation corresponding to the frequency band for which we desire to realize the determined frequency patterns. In order to establish the interval of approximation, it is necessary to know the parameters of the transmission long line. For a cable long line of diameter 0.8 mm we have

$$R \approx 70\,\Omega/\text{km}; \qquad C \approx 40\,nF/\text{km};$$

$$L \approx 0{,}7\,mH/\text{km}; \qquad G \approx 0.5\,\mu s/\text{km}.$$

If the interval $0 \leqslant X \leqslant b$ is an approximation interval, then, taking into account the dependence

$$X = l\sqrt{(R + j\omega L)(G + j\omega C)}\,,$$

we shall find that the upper limit of the frequency of the approximation band is the frequency f_0, which is approximately determined by the formula

$$f_0 \approx \frac{b^2}{l^2}\,\frac{1}{2\pi RC}\,. \tag{23}$$

Then, taking into consideration the numerical data quoted above, we shall obtain

$$f_0 \approx 5.68 \cdot 10^4 \frac{b^2}{l^2}. \tag{23a}$$

Expressing b as a function of l and f_0, by virtu of Formula (23a), we may write

$$b \approx l \sqrt{\frac{f_0}{5.68}}\, 10^{-2}. \tag{24}$$

We note from formula (23) that the limit frequency f_0 decreases with the square of the length of the long line being realized.

If we assume that the length of the long line is $l = 15$ km, the limit frequency is $f_0 = 6000\ H_z$, then we have:

$$b \approx 15 \cdot 10^{-2} \sqrt{\frac{6 \cdot 10^3}{5.68}} = 4.87 \approx 5. \tag{25}$$

For this case, the interval $0 \leqslant X \leqslant 5$ should be an approximation interval.

Let us turn now to the problem of approximation. The simplest manner of representing the function e^x by means of a rational function is to express e^x as the quotient of the functions $e^{\frac{x}{2}}$ and $e^{-\frac{x}{2}}$ and to expand these functions into Maclaurin's series

$$x^x = \frac{e^{\frac{x}{2}}}{e^{-\frac{x}{2}}} = \frac{1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \dots}{1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 - \dots} = \frac{P(x)}{Q(x)}. \tag{26}$$

This kind of approximation is not, however, very accurate. The approximation error for $|x| = 2$ amounts in this case to around 7 percent, and it grows rapidly with the increase in the variable x. We shall therefore apply a different kind of approximation.

Let us consider the rational function

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)}, \tag{27}$$

where $U_{\mu,\nu}$ and $V_{\mu,\nu}$ are the following polynomials of the variable x

$$\begin{aligned} U_{\mu,\nu}(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + a_\nu x\ , \\ V_{\mu,\nu}(x) &= \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_\mu x^\mu. \end{aligned} \tag{28}$$

Padé's approximation is called the approximation of the analytical function

$$B(x) = C_0 + C_1 X + C_2 X^2 + C_3 X^3 + \dots \tag{29}$$

by means of a rational function of the form (27) in such a manner that the expansion of the function (27) into a power series should contain $\mu + v$

successive coefficients identical with the coefficients $C_0, C_1, C_2, C_3 \ldots$ of the expansion of the function $B(x)$.

It can easily be observed that the condition of identity of the coefficients referred to involves the condition of satisfying the following equations

$$\left.\begin{aligned} &C_0 \beta_0 = \alpha_0, \\ &C_1 \beta_0 + C_0 \beta_1 = \alpha_1, \\ &\ldots\ldots\ldots\ldots \\ &C_\nu \beta_0 + C_{\nu-1} \beta_1 + \ldots + C_{\nu-\mu} \beta_\mu = \alpha_\nu, \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} &C_{\nu+1} \beta_0 + C_\nu \beta_1 + \ldots + C_{\nu-\mu+1} \beta_\mu = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &C_{\nu+\mu} \beta_0 + C_{\nu+\mu-1} \beta_1 + \ldots + C_\nu \beta_\mu = 0. \end{aligned}\right\} \tag{31}$$

The rational function (27 can of course be written in the form of the quotient of two polynomials $P_{\mu,\nu}(X)$, $Q_{\mu,\nu}(X)$ which are relatively prime (i.e. have no common divisors):

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} = \frac{P_{\mu,\nu}(x)}{Q_{\mu,\nu}(x)}.$$

It is possible to prove that for an arbitrary function $B(x)$, which is analytical near the point 0, there is one and only one pair of relatively prime polynomials approximating the function $B(x)$ in the sense of Padé, and that

$$\lim_{\mu+\to\infty} \frac{P_{\mu,\nu}(x)}{Q_{\mu,\nu}(x)} = B(x). \tag{32}$$

We shall now determine the rational function approximating the exponential function e^x in the sense of Padé. For this purpose, in accordance with [1] we shall consider the following integral

$$\int_0^1 e^{tx} F(t)\, dt,$$

where $F(t)$ is a given function differentiable up to the order m inclusively, and possessing derivatives, starting with the order $m+1$, equal as identities to zero

$$F^{(m+1)}(t) = F^{(m+2)}(t) = \ldots = 0.$$

Thus, we may write

$$\int_0^1 e^{tx} F(t)\, dt = \frac{F(1)}{x} e^x - \frac{F(0)}{x} - \frac{1}{x} \int_0^1 e^{tx} F'(t)\, dt.$$

Proceeding so m times, we shall obtain

$$\int_0^1 e^{tx} F(t)\,dt = e^x \left[\frac{F(1)}{x} - \frac{F(1)}{x^2} + \ldots + (-1)^m \frac{F^{(m)}(1)}{x^{m+1}} \right] -$$

$$- \left[\frac{F(0)}{x} - \frac{F'(0)}{x^2} + \ldots + (-1)^m \frac{F^{(m)}(0)}{x^{m+1}} \right].$$

And multiplying both sides of the equation by the factor $(-1)^m X^{m+1}$, we shall find

$$e^x [F^{(m)}(1) - F^{(m-1)}(1) X + \ldots + (-1)^m F(1) X^m] -$$

$$- [F^{(m)}(0) - F^{(m-1)}(0) X + \ldots + (-1)^m F(0) X^m] = (-1)^m X^{m+1} \int_0^1 e^{tx} F(t)\,dt.$$

Assume now that

$$F(t) = t^\mu (1-t)^\nu \tag{33}$$

and that $m = \mu + \nu$. Consequently we shall obtain

$$F(0) = 0, \quad F'(0) = 0, \ldots F^{(\mu-1)}(0) = 0, \quad F^{(\mu)}(0) \neq 0;$$

$$F(1) = 0, \quad F'(1) = 0, \ldots F^{(\nu-1)}(0) = 0, \quad F^{(\nu)}(1) \neq 0.$$

Therefore

$$e^x [F^{(\mu+\nu)}(1) - F^{(\mu+\nu-1)}(1) x + \ldots + (-1)^\mu F^{(\nu)}(1) x^\mu] -$$

$$- [F^{(\mu+\nu)}(0) - F^{(\mu+\nu-1)}(0) x + \ldots + (-1) F^{(\mu)}(0) x^\nu] =$$

$$= (-1)^{\mu+\nu} x^{\mu+\nu+1} \int_0^1 e^{tx} F(t)\,dt.$$

If we expand the right-hand side of the above equation into Maclaurin's power series, then the first term of such an expansion will be a term with the power $\mu + \nu + 1$.

Accordingly, the expansion of the function

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} = \frac{F^{(\mu+\nu)}(0) - F^{(\mu+\nu-1)}(0) x + \ldots + (-1)^\nu F^{(\mu)}(0) x^\nu}{F^{(\mu+\nu)}(1) - F^{(\mu+\nu-1)}(1) x + \ldots + (-1)^\mu F^{(\nu)}(1) x^\mu} \tag{34}$$

into Maclaurin's series will contain $\mu + \nu$ first successive coefficients identical with $\mu + \nu$ first successive coefficients in the expansion of the function e^x.

The function (34) then approximates the function e^x in the sense of Padé.

We shall now determine the coefficients of the rational function (34), we shall find[2)]

[2)] It should be borne in mind that $\binom{m}{n} = \frac{m!}{n!\,(m-n)!}$

$$F^{(n)}(t) = \sum_{\lambda=0}^{n} \binom{n}{\lambda}\binom{\mu}{\lambda} \lambda!\, t^{\mu-\lambda} \binom{\nu}{n-\lambda} (n-\lambda)!\,(1-t)^{\nu-n+\lambda}(-1)^{n-\lambda},$$

and then for

$$F^{(n)}(0) = \binom{n}{\mu}\mu!\binom{\nu}{n-\gamma}(n-\mu)!\,(-1)^{n-\mu} = (-1)^{n-\mu}\, n! \binom{\nu}{n-\mu},$$

and for

$$F^{(n)}(1) = \binom{n}{n-\nu}\binom{\mu}{n-\nu}(n-\nu)!\,(-1)^{\nu} = (-1)^{\nu}\, n! \binom{\mu}{n-\nu}.$$

Accordingly, the function approximating the exponential function e^x in the sense of Padé, can be written as follows

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} =$$

$$= \frac{(\mu+\nu)! + (\mu+\nu-1)!\binom{\nu}{1}x + (\mu+\nu-2)!\binom{\nu}{2}x^2 + \dots + \mu!\binom{\nu}{\nu}x^{\nu}}{(\mu+\nu)! + (\mu+\nu-1)!\binom{\mu}{1}x + (\mu+\nu-2)!\binom{\mu}{2}x^2 + \dots + (-1)^{\mu}\nu!\binom{\mu}{\mu}x^{\mu}}. \tag{35}$$

The polynomials $P_{\mu,\nu}(X)$ and $Q_{\mu,\nu}(X)$ in this case take the form

$$P_{\mu,\nu}(x) = 1 + \frac{\nu}{\mu+\nu}\frac{x}{1!} +$$

$$+ \frac{\nu(\nu-1)}{(\mu+\nu)(\mu+\nu-1)}\frac{x^2}{2!} + \dots + \frac{\nu(\nu-1)\dots 2.1}{(\mu+\nu)(\mu+\nu-1)\dots(\mu+1)}\frac{x^{\nu}}{\nu!}$$

$$Q_{\mu,\nu}(x) = 1 - \frac{\mu}{\mu+\nu}\frac{x}{1!} + \tag{36}$$

$$\frac{\mu(\mu-1)}{(\mu+\nu)(\mu+\nu-1)}\frac{x^2}{2!} + \dots + (-1)^{\mu}\frac{\mu(\mu-1)\dots 2.1}{(\mu+\nu)(\mu+\nu-1)\dots(\nu+1)}\frac{x^{\nu}}{\mu!}.$$

In accordance with the assumption, these polynomials satisfy the condition

$$\lim_{\mu+\nu\to\infty} \frac{P_{\mu,\nu}(x)}{Q_{\mu,\nu}(x)} = e^x. \tag{37}$$

It can be observed that the following relation holds

$$Q_{\mu,\nu}(x) = P_{\nu,\mu}(-x).$$

We are interested above all in the case in which the polynomials $P_{\mu}, Q_{\mu,\nu}$ are of the same degree — that is, the case for which $\mu = \nu$. Let us denote $\mu = \nu = n$, and $P_{\mu,\nu}(x) = P(x), Q_{\mu,\nu}(x) = Q(x)$.

We shall obtain

$$Q(x) = P(-x)$$

and

$$\left.\begin{aligned} P(x) &= 1 + \frac{1}{2}\frac{x}{1!} + \frac{n-1}{2(2n-1)}\frac{x^2}{2!} + \ldots + \frac{(n-1)!}{2(2n-1)\ldots(n+1)}\frac{x^n}{n!}, \\ Q(x) &= 1 - \frac{1}{2}\frac{x}{1!} + \frac{n-1}{2(2n-1)}\frac{x^2}{2!} - \ldots + (-1)^n\frac{(n-1)!}{2(2n-1)\ldots(n+1)}\frac{x^n}{n!}. \end{aligned}\right\} \qquad (38)$$

Assuming then the approximation in the sense of Padé, we shall have the coefficients in Formula (6), as follows

$$\left.\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{1}{2}, \\ a_2 &= \frac{1}{2}\frac{n-1}{2n-1}\frac{1}{2!}, \\ a_3 &= \frac{1}{2}\frac{(n-1)(n-2)}{(2n-1)(2n-2)}\frac{1}{3!}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n &= \frac{1}{2}\frac{(n-1)!}{(2n-1)\ldots(n+1)}\frac{1}{n!}. \end{aligned}\right\} \qquad (39)$$

In a particular case, when $n = 4$, we have

$$\left.\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{1}{2}, \\ a_2 &= \frac{3}{14}\frac{1}{2!} = \frac{3}{28}, \\ a_3 &= \frac{1}{14}\frac{1}{3!} = \frac{1}{84}, \\ a_4 &= \frac{1}{70}\frac{1}{4!} = \frac{1}{1680}. \end{aligned}\right\} \qquad (40)$$

We have, then, for this case the following rational function approximating the exponential function

$$e^x = \frac{1 + \frac{1}{2}x + \frac{3}{28}x^2 + \frac{1}{84}x^3 + \frac{1}{1680}x^4}{1 - \frac{1}{2}x + \frac{3}{28}x^2 - \frac{1}{84}x^3 + \frac{1}{1680}x^4} = \frac{P(x)}{Q(x)}. \qquad (41)$$

Below is given a table indicating the degree of approximation of the function $\frac{P(x)}{Q(x)}$ to e^x for different values of the variable x.

x	1	2	3	4	5
$\frac{P(x)}{Q(x)}$	2.7184	7.3887	20.0655	53.7275	128.616
e^x	2.7182	7.3891	20.086	54.598	148.41
$\Delta = \left\lvert e^x - \frac{P}{Q} \right\rvert$	0.000	0.000	0.02043	0.8705	19.794
δ%	0.00...	0.00...	0.10...	1.54...	13.33...

It can easily be verified that the coefficients (40) satisfy the inequality (22)

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 < 0,$$

and therefore, applying the above approximation, we arrive at the realizble impedances $Z_1^{(3)}$, $Z_1^{(4)}$, $Z_2^{(3)}$, $Z_2^{(4)}$.

The degree of approximation can be considerably increased by the application of a number of sections of the type X connected in a chain. The function e^x can thus be representend in the form of the product

$$e^x = \underbrace{e^{\frac{x}{N}} e^{\frac{x}{N}} \dots e^{\frac{x}{N}}}_{N} = e^{\left(\frac{x}{N}\right)N}$$

and then, approximating each of the N factors $e^{\frac{x}{N}}$ by the function of the form (41), we shall obtain

$$e^x = \left[\frac{1 + \frac{1}{2}\frac{x}{N} + \frac{3}{28}\left(\frac{x}{N}\right)^2 + \frac{1}{84}\left(\frac{x}{N}\right)^3 + \frac{1}{1680}\left(\frac{x}{N}\right)^4}{1 - \frac{1}{2}\frac{x}{N} + \frac{3}{28}\left(\frac{x}{N}\right)^2 - \frac{1}{84}\left(\frac{x}{N}\right)^3 + \frac{1}{1680}\left(\frac{x}{N}\right)^4}\right]^N = \left[\frac{P\left(\frac{x}{N}\right)}{Q\left(\frac{x}{N}\right)}\right]^N \tag{42}$$

In this manner, the approximation interval is increased N times.

In order to realize the function $\left[\frac{P}{Q}\right]^N$ we make use of a system of N sections of the type X in a chain connection, the transfer function being determined by the formula

$$e^{-N\Gamma} = \left[\frac{1-\sqrt{\frac{Z_1}{Z_2}}}{1+\sqrt{\frac{Z_1}{Z_2}}}\right]^N.$$

We then have

$$\frac{1-\sqrt{\frac{Z_1}{Z_2}}}{1+\sqrt{\frac{Z_1}{Z_2}}} = \frac{P\left(\frac{x}{N}\right)}{Q\left(\frac{x}{N}\right)}. \tag{43}$$

4. SYNTHESIS OF AN AMPLIFIER WITH NEGATIVE IMPEDANCE

Let us discuss a case in which an amplifier with negative impedance (negistor) is connected to the output of a transmission line having the length l, which is characterized by the wava impedence Z_T and the composite transfer coefficient Γ_T (Fig. 5).

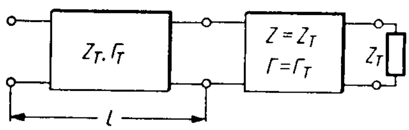

Fig. 5. Negator working in conjunction with a transmission long line

We shall call an optimal negistor working in conjunction with a cable long line, such a four-pole which is active, symmetric with respect to energy, and whose wave impedances — the initial and the secondary impedance — are equal to the wave impedance of the long line

$$Z = Z_T \tag{44}$$

and whose composite transfer coefficient in a given frequency band is equal as regards the modulus, and inverse as regards the sign of the wave composite transfer coefficient of the long line

$$\Gamma = -\Gamma_T; \quad \omega_1 \leqslant \omega \leqslant \omega_2. \tag{45}$$

If the conditions (44) and (45) are satisfied, the total attenuation of the entire system shown in Fig. 5 will be equal to zero, and moreover, there will occur no reflections of energy at the input and at the output of the active four-pole.

It should be noted that the assumption of the equality (45) over the entire frequency band would lead to a condition physically unrealizable, since a transmission line is characterized by the delays in electric waveforms. If, however, the equality (45) is satisfied in a finite frequency band, this is an agreement with the condition of physical realizibility of systems, provided that the system being designed is an active one. The problem of designing a negistor on the basis of the conditions (44) and (45), is very close to the problem of designing an artificial long line. As a basis of our considerations, we shall take the lattice four-pole shown in Fig. 1.

The transfer function of this four-pole is determined by the formula

$$e^{-\Gamma} = \frac{1 - \sqrt{\dfrac{Z_1}{Z_2}}}{1 + \sqrt{\dfrac{Z_1}{Z_2}}}.$$

Satisfying the condition (45), we assume that

$$\frac{1 - \sqrt{\dfrac{Z_1}{Z_2}}}{1 + \sqrt{\dfrac{Z_1}{Z_2}}} = \frac{P}{Q}, \tag{46}$$

where P and Q are polynomials defined by Formula (6). From Eq. (46), we shall obtain

$$\sqrt{\frac{Z_1}{Z_2}} = \frac{Q - P}{Q + P}. \tag{47}$$

Hence

$$Z_1 = \frac{Q - P}{Q + P} Z, \qquad Z_2 = \frac{Q + P}{Q - P} Z. \tag{48}$$

Therefore, assuming that $Z = Z_\Gamma$, and thus satisfying the condition (44), we shall obtain

$$Z_1 = -\frac{a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{n-1} x^{n-1}}{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_n x^n} Z_T, \tag{49}$$

$$Z_2 = -\frac{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_n x^n}{a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{n-1} x^{n-1}} Z_T. \tag{50}$$

It is now clear that the impedances Z_1 and Z_2 are (with the assumption $a_i > 0$) determined by negative functions. These impedances can be realized by means of tube or transistor convertors. Let us now assume that the impedance Z_1 is realized in an arc circuit and the impedance Z_2 — in a dynatron circuit. Let the coefficient of conversion be denoted by β_k. Then

$$Z_1 = \beta_k Z_\alpha; \qquad Z_2 = \frac{Z_\beta}{\beta_k}$$

and

$$Z_\alpha = \frac{Z_1}{\beta_k}; \qquad Z_\beta = \beta_k Z_2. \tag{51}$$

If for a band being amplified $w_1 \leqslant w \leqslant w_2$ we assume that

$$\beta_k \approx -1, \tag{52}$$

then for this band we shall obtain precisely:

$$\begin{aligned} Z_\alpha &= -Z_1, \\ Z_\beta &= -Z_2. \end{aligned} \tag{53}$$

Accordingly, in agreement with Eqs. (49) and (50) the impedances Z_α and Z_β may be designed in terms of the following formulae

$$Z_\alpha = \frac{a_1\sqrt{\frac{Z_a}{Z_b}} + a_3\left(\sqrt{\frac{Z_a}{Z_b}}\right)^3 + \ldots + a_{n-1}\left(\sqrt{\frac{Z_a}{Z_b}}\right)^{n-1}}{a_0 + a_2\frac{Z_a}{Z_b} + \ldots + a_n\left(\frac{Z_a}{Z_b}\right)^{\frac{n}{2}}}\sqrt{Z_a Z_b} \tag{54a}$$

$$Z_\beta = \frac{a_0 + a_2\frac{Z_a}{Z_b} + \ldots + a_n\left(\frac{Z_a}{Z_b}\right)^{\frac{n}{2}}}{a_1\sqrt{\frac{Z_a}{Z_b}} + a_3\left(\sqrt{\frac{Z_a}{Z_b}}\right)^3 + \ldots + a_{n-1}\left(\sqrt{\frac{Z_a}{Z_b}}\right)^{n-1}}\sqrt{Z_a Z_b} \tag{54b}$$

For the case $n = 4$ we shall have:

$$\begin{aligned} Z_\alpha &= \frac{a_1 Z_a Z_b^2 + a_3 Z_a^2 Z_b}{a_0 Z_b^2 + a_2 Z_a Z_b + a_4 Z_a^2}, \\ Z_\beta &= \frac{a_0 Z_b^2 + a_2 Z_a Z_b + a_4 Z_a^2}{a_1 Z_a Z_b^2 + a_3 Z_a^2 Z_b}. \end{aligned} \tag{55}$$

The structures of the two-poles Z_α and Z_β are given in Fig. 3. The two-poles Z_α and Z_β are then designed in a manner similar to the two-poles Z_1 and Z_2 of the artificial long line, using the approximation in the sense of Padé. Using a rational function with polynomials of the fourth degree, $n = 4$, we obtain a negistor compensating the attenuation of the transmission line of length $l = 15$ km in the frequency band up to 6 kHz.

Let us now consider the stability of a negistor designed in the above manner

$$e^{-\Gamma} = \frac{1 - \sqrt{\frac{Z_1}{Z_2}}}{1 + \sqrt{\frac{Z_1}{Z_2}}}.$$

Making use of Formula (51), we shall obtain the relation

$$\frac{Z_1}{Z_2} = \beta_k^2 \frac{Z_\alpha}{Z_\beta}. \tag{56}$$

It should be noted that the impedances Z_α and Z_β are designed in terms of Formulae (52a) and (52b). We can therefore write

$$Z_\alpha = \frac{a_1 x + a_3 x^3 + \dots}{a_0 + a_2 x^2 + \dots} Z,$$

$$Z_\beta = \frac{a_0 + a_2 x^2 + \dots}{a_1 x + a_3 x^3 + \dots} Z$$

and consequently

$$\sqrt{\frac{Z_1}{Z_2}} = \frac{a_1 x + a_3 x^3 + \dots}{a_0 + a_2 x^2 + \dots} \beta_k.$$

Denoting, then

$$\frac{a_1 x + a_3 x^3 + \dots}{a_0 + a_2 x^2 + \dots} = K \tag{57}$$

we may write the transfer function of a negistor as

$$e^{-\Gamma} = \frac{1 - K\beta_k}{1 + K\beta_k}. \tag{58}$$

The stability of the system is dependent on the equation

$$1 + K(s)\,\beta_k(s) = 0. \tag{59}$$

If the roots of this equation contain negative real parts, the system under consideration is not a stable system. Otherwise, the system is stable. The investigation of stability may be effected in terms of the frequency criteria. From the expression (58), it follows directly that a system is stable when the amplitude-phase characteristic of the function

$$K(j\omega)\,\beta_k(j\omega)$$

does not comprise the point $-1 + j0$ [3].

[3] By virtue of Nyquist's criterion of stability.

A negistor may most conveniently be realized in the symmetric system shown in Fig. 6. The amplification band can be extended by means of connecting a number of sections into a chain. It should, however, be noted that there is a difficulty in realizing a negistor with a wide amplification band consisting in the design of suitable convertors which realize negative impedances in a wide frequency band. Moreover, the stability of the system may be an obstacle in realizing a wide-band negistor. We can exert an influence on stability by means of properly shaping the pattern of the conversion coefficient $\beta_k(j\omega)$. In systems encountered in engineering, the conversion coefficient beyond the amplification band changes its sign, and consequently the impedances in the range of high frequencies become positive.

We have the following property for a system with the transfer function (58) to be a stable system it is a sufficient condition to satisfy the inequality

$$|K\beta_k| < 1 \quad \text{for} \quad 0 \leqslant \omega \leqslant \omega_1, \tag{60}$$

where ω_1 is the angular frequency for which $\mathcal{Re}\,\beta_k$ changes its sign from a minus to a plus. The above property is the result of Nyquist's criterion.

The condition (60) can be weakened by imposing a requirement that $K_{\beta\,k}$ be smaller than unity only near a frequency for which arg $K\beta_k = \pi$.

In the above reasoning, we have assumed that the component $\mathcal{Re}\,\beta_k$ is negative, starting with the zero frequency up to ω_1. Actually $\mathcal{Re}\,\beta_k$ in the range of very low frequencies is a positive quantity. This fact, however, has no bearing on the result of the reasoning effected above. According to the property which has been proved, in order to state the stability of a negistor, it is necessary to investigate the plot of the modulus $|K\beta_k| = \left|\sqrt{\frac{Z_\alpha}{Z_\beta}}\,\beta_k\right|$ in the frequency band from 0 to the limit frequency ω_1, for which $\mathcal{Re}\,\beta_k$ changes its sign and the impedances Z_1 and Z_2 become positive. The negistor will certainly be stable, if the modulus $K\,\beta_k$ is smaller than unity near the frequency for which arg $K\,\beta_k = \pi$. In designing convertors, we should tend to have the conversion coefficients in the amplification band real, negative and approximately constant. For only then can we successfully realize the impedances Z_1 and Z_2. The impedces Z_1 and Z_2 would have been realizable in the most appropriate manner, if in the amplification band

$$\beta_k = -1.$$

However, to ensure the stability of the system, we may assume that in the amplification band $|\beta_k|$ is smaller than unity, for instance $\beta_k = -0.9$. The plot of the function $K = \sqrt{\frac{Z_\alpha}{Z_\beta}}$ is shown in Fig. 7. It can be observed that this function comprises the point $1 + j0$, whereas the function $0.9\ K$ dose not comprise the point $1 + j0$. Hence, we may conclude that a negistor is certainly a stable system in the case in which the conversion coefficient β_k in the amplification band is equal to 0.9. Making

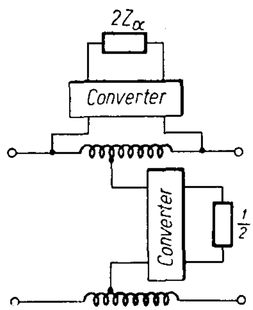

Fig. 6. Basic diagram of a negistor

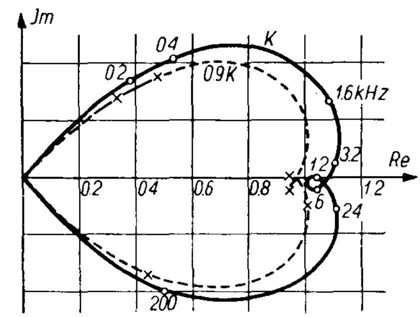

Fig. 7. The function $K = \sqrt{\frac{Z_\alpha}{Z_b}}$ and $0.9\,K$

use of the plot of the function K given in Fig. 7, we can determine graphically, on the basis of Formula (58), the transfer function and the amplification of the negistor. Fig. 8 shows the functions $1 + K$ and $1 - K$, which are useful in determining graphically the function $e^{-\Gamma}$. In working out the

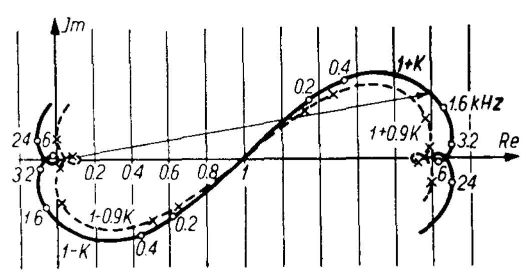

Fig. 8. Auxiliary functions necessary for graphical determination of the transfer function of a negistor

diagrams presented in the Figs mentioned above, we have assumed that the inductivity of the long line equals zero, which is fully admissible. Fig. 10 presents the amplification of a negistor, as a calculated by the graphical method for two cases — namely, the case in which in the amplification band the coefficient $\beta_k = -1$, and that in which $\beta_k = -0.9$. The diagrams in this Fig. indicate how great is the influence of convertors on the frequency pattern of the amplification due to the negistor.

Two frequency scales are given in Fig. 9. The upper scale corresponds to the case in which the length of the long line is $l = 15$ km, and the line is amplified by a negistor consisting of one section. The lower scal corresponds to the case of a two-section negistor working in conjunction with a long line $l = 15$ km in length.

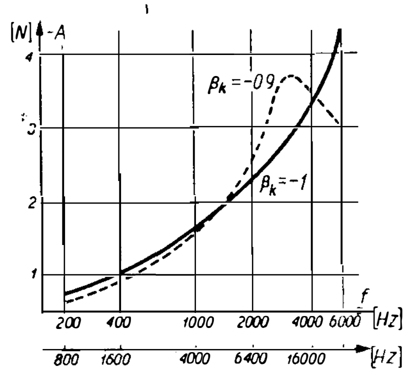

Fig. 9. Amplification of a negistor for two cases: $\beta_k = -1$, and $\beta_k = -0.9$

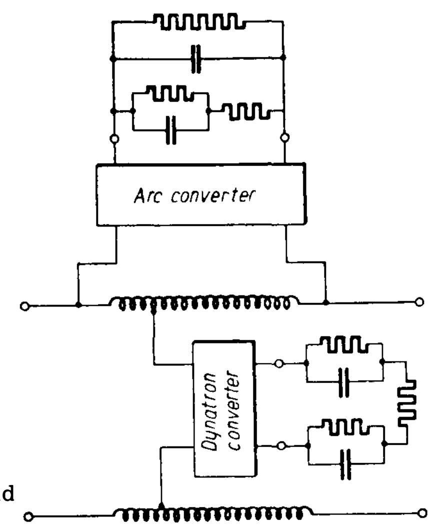

Fig. 10. Sections of a negistor and the structure of equilizers

It can be noted that, for the latter case, in a band up to 6400 Hz, the error brought about by the substitution of the coefficient $\beta_k = -0.9$ for $\beta_k = -1$ does not exceed ± 0.1 N. The correction of the plot $K\beta_k$ by means of a decrease in the coefficient β_k is the simplest manner of correction. The error can be considerably lowered if a more complicated correction is applied to the frequency pattern for the conversion coefficient β_k. In order to obtain the optimal solution, we should tend to have $|\beta_k| \approx 1$ in the amplification band, and beyond the amplification band the modulus of the coefficient β_k should decrease rapidly in the range up to the frequency ω_1 for which $R_e \beta_k$ changes its sign from a minus to a plus. We are not interested in the further part of the characteristic of β_k coefficient, since it exerts no influence on the stability of the system.

A section of the negistor, and the structures of the two-poles Z and Z, are shown in Fig. 10. The values of the elements of these two-poles are calculated on the basis of Formulae (21) with the assumption that

$$Z_a = lR,$$
$$Z_b = \frac{1}{l(G + SC)}, \tag{61}$$

where R, C, G are unit parameters of the long line and l is the length of the long line in kilometers.

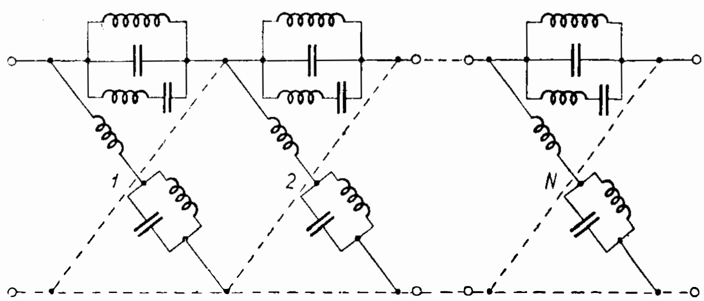

Fig. 11. Delaying long line

5. FINAL REMARKS

The method presented in the paper can be used, further, in the design of a delaying four-pole. The transfer function of an ideal delaying four-pole takes the form of the exponential function

$$K(s) = e^{-\lambda s} \tag{62}$$

where λ is the parameter possessing the dimension of time, and determining the delay brought about by the four-pole.

Assuming that the wave impedances of a delaying four-pole are real and equal to R

$$R = \sqrt{Z_a Z_b}\,, \tag{63}$$

and that

$$x = \sqrt{\frac{Z_a}{Z_b}} = \lambda s\,. \tag{64}$$

we shall obtain from Eqs. (63) and (64)

$$Z_a = \lambda R s = L s\,, \tag{65}$$

where

$$L = \lambda R\,, \tag{66}$$

and

$$Z_b = \frac{R}{\lambda s} = \frac{1}{Cs}\,, \tag{67}$$

where

$$C = \frac{\lambda}{R}. \tag{68}$$

According to the relations (65) and (67), the impedance Z_a is the inductivity, and the impedance Z_b — the capacity. The values of the inductivity L and the capacity C are given by Formulae (66) and (68), respectively.

The diagrams shown in Fig. 3 determine the structure of the two-poles Z_1 and Z_2 of a lattice four-pole. The values of the particular elements of these two-poles are calculated from Eqs. (21), with the assumptions that $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{3}{28}$, $a_3 = \frac{1}{84}$, $a_4 = \frac{1}{1680}$.

The degree of accuracy in the realization of a delaying four-pole can be increased several times, if we connect a few or several lattice sections into a chain. In the case of a system with N sections, the inductivity L and the capacity C may be calculated from the formulae

$$L = \frac{\lambda}{N} R, \qquad C = \frac{\lambda}{N} \frac{1}{R}. \tag{69}$$

The basic diagram of an N-section delaying four-pole designed by the method described is shown in Fig. 11.

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[1] Perron, O., — Die Lehre von den Kettenbruchen, Chelsea Publishing Company, New York, 1950.

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[3] Zobel, O. J., — Distortion correction in electrical circuits with constant resistance recurrent networks. The Bell System Technical Journal, July, 1928.

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Pagination in the table of contents should be changed to read as follows:

Item	From	To
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ERRATA

Page	Line Top	Line Bottom	Reads	Should read
11		8	$(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\min(1,\|x-y\|_k)$	$\varrho(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\min(1,\|x-y\|_k)$
28		12	$e^{\lambda g(x)}=\frac{p}{p-\lambda}.$	$e^{\lambda g(x)}=\frac{p}{p-\lambda}(1).$
28		1	$tg^{\lambda}\left(\frac{x}{2}+\frac{\pi}{4}\right)=\frac{p}{p-\lambda}.$	$tg^{\lambda}\left(\frac{x}{2}+\frac{\pi}{4}\right)=\frac{p}{p-\lambda}(1).$
29	9		$\frac{1}{p-\lambda}=T[e^{\lambda g(x)}]=\frac{1}{\lambda}[e^{\lambda g(x)}-1],$	$\frac{1}{p-\lambda}(1)=T[e^{\lambda g(x)}]=\frac{1}{\lambda}[e^{\lambda g(x)}-1],$
29	11		$\frac{p^{-1}}{p-\lambda}=\frac{1}{y^2}[e^{\lambda g(x)}-\lambda g(x)-1].$	$\frac{p^{-1}}{p-\lambda}(1)=\frac{1}{y^2}[e^{\lambda g(x)}-\lambda g(x)-1].$
29		9	$y^{-\alpha}y''-\alpha y^{-\alpha-1}y'^1=\frac{1}{1-\alpha}p^2y^{1-\alpha}$	$y^{-\alpha}y''-\alpha y^{-\alpha-1}y'^2=\frac{1}{1-\alpha}p^2y^{1-\alpha}$
33	11		$T(x)t=\begin{cases}\int_1^t\frac{x(\tau)}{\tau}d\tau & \text{for} \quad \tau\geqslant 1\\ 0 & \text{for} \quad \tau<1\end{cases}$	$Tx(t)=\begin{cases}\int_1^t\frac{x(\tau)}{\tau}d\tau & \text{for} \quad t\geqslant 1,\\ 0 & \text{for} \quad t<1.\end{cases}$
33		16	$t^n x^{(n)}=p(p-1)\ldots$	$t^n x^{(n)}(t)=p(p-1)\ldots$
44	3		$\hat{a}=\hat{0}$ or $\hat{b}-\hat{0}.$	$\hat{a}=\hat{0}$ or $\hat{b}=\hat{0}.$
46	6		$p\left\{{a \atop n+k}\right\}-pa_k=\{a_{n+k+1}\},$	$p\{a_{n+k}\}-pa_k=\{a_{n+k+1}\},$
121		15	$k\sum_{n=1}^{\infty}k_{\gamma}[n,\varepsilon]e^{-ns};$	$k\sum_{n=1}^{\infty}h_{\gamma}[n,\varepsilon]e^{-ns};$
126	11		$x_{2\,ust}[n,\varepsilon]=A_0(\varepsilon)e^{j\omega n}$	$x_{2\,steady}[n,\varepsilon]=A_0(\varepsilon)e^{j\bar{\omega} n}.$
131	2		$X^*_{wy}(z,\varepsilon)+[K^*_{i_1}(z,\varepsilon)+K^*_{i_2}(z,\varepsilon)+\ldots+K^*_{i_N}(z,\varepsilon)]X^*_{we}(z)$	$X^*_{out}(z,\varepsilon)+[K^*_{i_1}(z,\varepsilon)+K^*_{i_2}(z,\varepsilon)+\ldots+K^*_{i_N}(z,\varepsilon)]X^*_{in}(z)$
142		6	$X^*_{wy}(z,\varepsilon)=$	$X^*_{out}(z,\varepsilon)=$
218	16		Left out	$w\geqslant\frac{n+4}{3}+1. \quad (50)$